ELEMENTARILY EQUIVALENT FIELDS
WITH INEQUIVALENT PERFECT CLOSURES

CARLOS R. VIDELA

Abstract. We give a counterexample to the following conjecture due to L. V. den Dries: Let $F, L$ be two fields of characteristic $p$. If $F \equiv L$ then $F_{1/p}^\infty \equiv L_{1/p}^\infty$.

Introduction. Van den Dries conjectures in [3, Stellingen 4] that elementarily equivalent fields have elementarily equivalent perfect closures. We will give a counterexample using a predicate introduced by Cherlin [1] in the study of definability in power series fields of nonzero characteristic.

Our counterexample is as follows: Fix a prime number $p$. Let $\mathbb{F}_p$ be the algebraic closure of the prime field $\mathbb{F}_p$, let $K_0 = \mathbb{F}_p(t)$ be the rational function field, and let $L_0 = K_0^*$ be a nonstandard extension of $K_0$, e.g. an ultrapower of $K_0$. In particular $K_0 \equiv L_0$. We will prove:

Theorem. The perfect closures of $K_0$ and $L_0$ are not elementarily equivalent.

The idea behind the proof is the following: we will relate the $p^n$th powers of $t$, for $n$ an integer, to the $p^n$th roots of $t$ in the perfect closure $K = K_0^\infty$ of $K_0$. However the $p^N$th powers of $t$ with $N$ infinite will not correspond to roots of $t$ in $L = L_0^\infty$.

We will produce a sentence $\varphi$ whose meaning is approximately

"For all $n$ if $t^{p^n}$ exists then $t^{1/p^n}$ exists."

$\varphi$ will hold in $K$ and fail in $L$.

We make extensive use of the polynomial $\tau(x) = x^p - x$. We will need first order definitions of the following three predicates defined in $K$.

(1) $\text{Con}(x): "x \in \mathbb{F}_p^\ast$",

(2) $\text{Reg}(x): "\exists b \in \mathbb{F}_p^\ast \setminus \{0\}, bx \in \tau(K)$",

(3) $\text{Link}(y, x): "\exists n \in \mathbb{Z}, a \in \mathbb{F}_p^\ast, y = x^{p^n} + a \& \text{Reg}(x)$.”

For $\text{Con}(x)$ we can do the following: choose $n > 2$ and relatively prime to $p$. Then the curve $E$ defined by the equation $x^n + y^n = 1$ is nonrational [2, p. 7]. This shows that the only solutions to the equation in $K_0$ must be constants. The same is true of $K$: for if $x_0, y_0 \in K$ are such that $x_0^n + y_0^n = 1$, then by taking $p^k$th powers for $k$ large enough, we get

\[
(x_0^{p^k})^n + (y_0^{p^k})^n = 1
\]
with \( x_0^a, y_0^a \in K_0 \). Hence \( x_0 \) and \( y_0 \) are constants. We have shown that \( \text{Con}(x) \) defines the same set in \( K_0 \) and in \( K \), namely \( \mathbb{F}_p \). \( \text{Reg}(x) \), read "\( x \) is regular," is visibly first order definable. We will use the following fact: Let \( x \in K_0 \). Then \( K_0 = \text{Reg}(x) \iff K \models \text{Reg}(x) \). A proof of this is implicit in Lemma 1 below.

The definition of \( \text{Link}(y, x) \) is as follows: First define a predicate \( L_0(y, x) \) in the following way:

\[
\forall a \in \mathbb{F}_p \exists b \in \mathbb{F}_p \left( ay - bx \in \tau[K] \right).
\]

Notice that \( L_0(y, x) \) implies \( x \) is regular: take \( a = 0 \). When \( L_0(y, x) \) holds, we write \( y[a] \) for the element \( b \) satisfying \( ay - bx \in \tau[K] \), and we define \( \text{Link}(y, x) \) by a first order formalization of

\[
\text{"} L_0(y, x) \text{&} \forall a_1, a_2 \in \mathbb{F}_p \left( y[a_1a_2] = y[a_1]y[a_2] \right) \text{"}
\]

We will prove later that this has the intended meaning in \( K \). Notice also that \( \tau[K] \) is an additive abelian group and that, if \( L_0(y, x) \), then the map \( a \mapsto y[a] \) is an abelian group homomorphism (so that \( \text{Link}(y, x) \) says it is a field isomorphism too).

We can now write down the sentence \( \Phi \):

\[
\forall x, y \left( \left[ \text{Link}(y, x) \& \neg \text{Con}(y - x) \& \forall a (\text{Con}(a) \Rightarrow \text{Reg}(x - a)) \right] \Rightarrow \exists z, b \left[ \text{Con}(b) \& \text{Link}(z, x) \& \text{Link}(xz, x(y - b)) \& \neg \text{Con}(y - z) \right] \right).
\]

We will show that \( \Phi \) is true in \( K \); on the other hand taking \( x = t \) we will see that \( \Phi \) is false in \( L \). It turns out that the content of \( \Phi \) is approximately

\[
\forall x \forall n \neq 0 \forall y \in x^p^n + \mathbb{F}_p \left( \exists z, a \in \mathbb{F}_p \left( (xz)^p^n = x(y - a) \& x - z^p^n \in \mathbb{F}_p \right) \right).
\]

1. The interpretation of \( \Phi \) in \( K \).

**Lemma 1.** For \( x, y \in K_0 \), if \( K \models \text{Link}(y, x) \), then \( K_0 \models \text{Link}(y, x) \).

**Proof.** Let \( a \in \mathbb{F}_p \). Then there exist \( b \in \mathbb{F}_p \) and \( w \in K \) such that \( ay - bx = w^p - w \). Note that \( w \) is inseparable over \( K_0 \). Since \( w^p - w \in K_0 \), it follows that

\[
K_0(w) = K_0(w^p) = K_0(w^{p^2}) = \cdots \text{. Hence } w \text{ is separable over } K_0 \text{ and so must be in } K_0.
\]

We will show that \( \text{Link}(y, x) \) has the intended meaning by proving that the map \( a \mapsto y[a] \) is a first-order definable automorphism of \( \mathbb{F}_p \) with parameters in \( \mathbb{F}_p \). It will then follow that \( y[a] = a^p^n \) for some \( n \in \mathbb{Z} \). Finally, we show that this implies \( y - x^p^n \) is a constant.

For \( x \in K_0 \) let \( P_x \) be the set of all poles (finite and at infinity) \( \mu \) of \( x \). For each \( \mu \) let \( \mathbb{F}_p((t_\mu)) \) be the completion of \( K_0 \) with respect to the valuation \( V_\mu \) determined by \( \mu \). \( K_0 \) is embedded in \( \mathbb{F}_p((t_\mu)) \). We will make use of the following fact [1, p. 103]: Let \( x \in \mathbb{F}_p((t_\mu)) \), \( x = \sum x_i t_\mu^i \), then \( x \in \tau[\mathbb{F}_p((t_\mu))] \) if and only if the following conditions \( (c_i) \) for \( i < 0, i \neq 0 \text{ (mod } p) \) are satisfied:

\[
\sum_{n > 0} (x_{i,n} n^{1/p^n}) = 0.
\]

**Remark.** Notice that if \( \nu_\mu(x) > 0 \), then \( x \in \tau[\mathbb{F}_p((t_\mu))] \), since the left-hand side of \( (c_i) \) is zero.
Lemma 2. Suppose \( x, y \in K_0 \) and \( \text{Link}(y, x) \). Then the map \( a \to y[a] \) is a first order definable (with parameters) automorphism of \( \hat{F}_p \).

Proof. Fix \( \mathfrak{p} \in P_x \) and let \( a \in \hat{F}_p \). There exists \( b \in \hat{F}_p \) such that \( ay - bx \in \tau[K_0] \).

In the completion \( \hat{F}_p((t_\mathfrak{p})) \), the finitely many equations \( (c_i) \) define a first-order formula \( \theta_p(u, w, \tilde{c}_p) \), where \( \tilde{c}_p \) is a sequence of parameters (essentially the coefficients in the principal parts of the Laurent expansions of \( y \) and \( x \)), such that, for \( a \) and \( b \) as above \( \theta_p(a, b, \tilde{c}_p) \) holds.

Define \( \psi(u, w) = \Lambda_{\mathfrak{p} \in P_x} \theta_p(u, w, \tilde{c}_p) \). We claim that for any \( a \in \hat{F}_p \), \( b = y[a] \Leftrightarrow \psi(a, b) \). The claim follows if we show that given \( a \in \hat{F}_p \) there exist one and only one \( b \in \hat{F}_p \) with \( \psi(a, b) \). So suppose \( \psi(a, b) \) and \( \psi(a, b') \). This implies that \( (b - b')x \in \tau[\hat{F}_p((t_\mathfrak{p}))] \) for all \( \mathfrak{p} \in P_x \), hence \( (b - b')x \in \tau[K_0] \) and since \( x \) is regular, \( b = b' \).

For the existence, take \( b = y[a] \).

Lemma 3. Let \( F \) be an algebraically closed field of characteristic \( p \). Then

(i) if \( \phi: F \to F \) is an automorphism of \( F \) definable over \( F \), there exists \( r(x) \in F(x) \) and \( n \in \mathbb{N} \) such that, except for finitely many elements of \( F \), we have \( \phi(a) = r(a)^{1/p^n} \).

(ii) if \( \phi: F \to F \) is an automorphism of \( F \) and \( r(x) \in F(x) \) is such that \( r(a) = \phi(a) \) for all \( a \in F \) except for a finite set, then \( r(x) = x^{p^n} \) for some \( k \in \mathbb{Z} \).

Proof. Let \( t \) be transcendental over \( F \) and let \( \hat{F}(t) \) be the algebraic closure of \( F(t) \). Then \( F \prec \hat{F}(t) \). Let \( \tilde{\phi} \) be the definable extension of \( \phi \) to \( \hat{F}(t) \). We have \( \tilde{\phi} \in \text{Aut} \hat{F}(t) \).

Let \( G = \text{Gal}(\hat{F}(t)/F(t)) \), \( \psi(x, y, \tilde{c}) \) be a definition of \( \phi \) over \( F \). Then for any \( \sigma \in G \) we have

\[
\hat{F}(t) \models \psi(t, \tilde{\phi}(t), \tilde{c})
\]

\[
\Rightarrow \hat{F}(t) \models \psi(\sigma(t), \sigma(\tilde{\phi}(t)), \sigma(\tilde{c}))
\]

\[
\Rightarrow \hat{F}(t) \models \psi(t, \sigma(\tilde{\phi}(t)), \tilde{c}).
\]

Hence \( \sigma(\tilde{\phi}(t)) = \tilde{\phi}(t) \), i.e. \( \tilde{\phi}(t) \in F(t)^{1/p^n} \). This proves \( \tilde{\phi}(t) = r(t)^{1/p^n} \) for some \( r(x) \in F(x) \) and \( n \in \mathbb{N} \). Next we show that the set \( A := \{ a \in F \mid \tilde{\phi}(a) = r(a)^{1/p^n} \} \) has finite complement. This follows from the following two facts:

(a) \( A \) is infinite,

(b) \( F \) is strongly minimal.

To prove (a) note that for any finite set \( S \subseteq F \)

\[
\hat{F}(t) \models \exists x (\tilde{\phi}(x) = r(x)^{1/p^n} \& x \notin S),
\]

hence \( F \models \exists x (\tilde{\phi}(x) = r(x)^{1/p^n} \& x \notin S) \).

To prove (ii) note that for \( t \) transcendental (as above) we have infinitely many \( a \in F \) such that

\[
r(at) = r(a)r(t).
\]

Hence the set of zeros \( Z \) and the set of poles \( P \) of \( r(x) \) are closed under multiplication by \( a \) for infinitely many \( a \in F \). Since \( Z \) and \( P \) are finite sets we have \( Z = P = \{0\} \cup \{\infty\} \), and so \( r(x) = x^{p^n} \). ∎
It follows that $\phi(x) = r(x)$ for all $x \in F$.

**Corollary.** With the assumptions of Lemma 3 there exists $k \in \mathbb{Z}$ such that $y[a] = a^p^k$.

The following fact will be used in the proof of Proposition 1.

**Fact.** If $y, x \in K_0$ and $\text{Link}(y, x)$, then $P_y = P_x$.

**Proof.** By symmetry it is enough to show that $P_y \subseteq P_x$. Suppose not. Let $\mu \notin P_y - P_x$. Then, since $\nu_\mu(x) \geq 0$, for all $a \in \hat{F}_p$ we have $ay \in \tau([\hat{F}_p((\tau_a))])$. Since $\hat{F}_p$ is infinite, it follows by using equation $(c_\mu)$ that the principal part of the Laurent expansion of $y$ must be zero, hence $\nu_\mu(y) \geq 0$, a contradiction.

**Proposition 1.** Let $x, y \in K_0$ and assume that $x$ is regular. Then

$$K_0 \models \text{Link}(y, x) \iff \exists n \in \mathbb{Z} \text{ and } c \in \hat{F}_p \text{ such that } y = x^{p^n} + c.$$  

**Proof.** Assume $\text{Link}(y, x)$. The corollary to Lemma 4 implies that $y[a] = b = a^p^n$ for some $n \in \mathbb{Z}$. Hence $ay - a^p^n x \in \tau([\hat{F}_p((\tau_a))])$ for all $\mu \in P_x$ and all $a \in \hat{F}_p$. We claim that $\nu_\mu(y - x^{p^n}) \geq 0$ for all $\mu \in P_x$. Conditions $(c_\mu)$ yield the following

$$(c_\mu) \quad \sum_{k=0}^{N_0} (ay_{ip^k} - a^p^n x_{ip^k})^{1/p^k} = 0,$$

where $N_0 \in \mathbb{N}$ and the $y_{ip^k}, x_{ip^k}$ are the coefficients appearing in the principal part of the Laurent expansions of $y$ and $x$. Taking $p^{N_0}$-th powers in the above equation gives us a polynomial in $a$, which is identically zero. By distinguishing the possibilities $N_0 > n$, $N_0 = n$, and $N_0 < n$ one shows that $y_i^{p^n} = x_{ip^k}$ for all $i < 0$.

To conclude the proof note that a pole of $y - x^{p^n}$ is a pole of $y$ or $x$ and by the fact stated before the proposition it is a pole of $x$. Hence $y - x^{p^n}$ has no poles at all and so must be a constant. For the other implication take $b = a^p^n$; uniqueness is a consequence of the regularity of $x$.

**Corollary.** For $y, x \in K, x$ regular,

$$K \models \text{Link}(y, x) \iff \exists n \in \mathbb{Z} \text{ and } a \in \hat{F}_p \text{ such that } y = x^{p^n} + a.$$  

**Proof.** Taking $p^n$-th powers yields an automorphism of $K$. Therefore we may take $y, x \in K_0$ and note that regularity over $K$ or $K_0$ is the same. The result follows from Lemma 1 and Proposition 1.

**Proposition 2.** $K \models \Phi$.

**Proof.** Let $x, y \in K$ satisfy $\text{Link}(y, x)$. Then $y = x^{p^n} + a$ for some $n$ and $a$. We may assume $n \neq 0$ since otherwise $y - x$ is a constant. Take $b = a$ and $z = x^{p^n}$. Clearly $\text{Link}(z, x)$. Then $x(y - b) = xx^{p^n}$ and $xz = xx^{p^n} = (xx^{p^n})^{p^n}$. We have that $y - z$ is not constant and $\text{Link}(xz, x(y - b))$.

**2. The interpretation of $\Phi$ in $L$.** The set defined by $\text{Con}(x)$ in $L$ and in $L_0$ is the same; denote it by $\hat{F}_p^*$. Lemma 1 remains true for the pair $L_0, L$. By transfer applied to the structure $(L_0, \mathbb{Z}^*, \hat{F}_p^*)$ we get that, for $x, y \in L, x$ regular,

$$L \models \text{Link}(y, x) \iff \exists n \in \mathbb{Z}^* \text{ and } a \in \hat{F}_p^* \text{ such that } y = x^{p^n} + a.$$
PROPOSITION 3. $L \not\equiv \Phi$.

PROOF. Take $x = t$ and $y = t^{p^N}$ with $N$ infinite and positive. Clearly $x$ is regular and linked to $y$. Next, for any $a \in \hat{F}_p^*$, we claim that $t(t^{p^N} - a)$ is regular. Otherwise, by transfer we could have $bt(t^{p^N} - a') \in \tau[K_0]$ with $b \neq 0$, $b, a' \in \hat{F}_p^*$, and $n$ a positive integer. But this is impossible. So, assume there exists $z, a \in \hat{F}_p^*$ such that $\text{Link}(z, t)$ and $\text{Link}(zt, t(t^{p^N} - a))$. Then $z = t^{p^N} + a', a' \in \hat{F}_p^*$, $n \in \mathbb{Z}^*$.

Notice that if $n < 0$, then $n$ cannot be infinite since $L = \bigcup_{n \in \mathbb{Z}} L^{1/p^n}$.

From $\text{Link}(zt, t(t^{p^N} - a))$ we get an equation of the form

$$t(t^{p^N} + a') = (t(t^{p^N} - a))^{p^k} + a''$$

for some $a'' \in \hat{F}_p^*$ and $k \in \mathbb{Z}^*$. Note that $a''$ must be equal to zero. We show (1) cannot hold.

**Case 1.** $a' = 0$, (or $a = 0$). Then $a = 0$ and we have $t^{p^N+1} = t^{p^N}$, so $p^n + 1 = p^k + p^{N+k}$. The equation has the following solutions: (a) $k = 0$, $n = N$, and (b) $N + k = 0$, $n = k$. (a) implies $y = z$ which we have excluded and (b) implies $n = -N$ which is also impossible as $N$ is infinite.

**Case 2.** $a' \neq 0$ and $a \neq 0$. In this case $k = 0$ so $N = n$ and $y - z = a$ which is impossible. Thus $\Phi$ is false in $L$, and $K \not\equiv L$.

I thank G. Cherlin for his help in the preparation of this paper.

REFERENCES


DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903