SHORTER NOTES

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AN ESTIMATE FOR THE VARIANCE OF A BOUNDED MEASURABLE RANDOM VARIABLE

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ABSTRACT. An estimate is provided for the variance of a real-valued essentially bounded measurable random variable in terms of its ess sup and ess inf.

Let \( I^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, \ i = 1, 2, \ldots, n \} \) and let \( \mu \) be the usual Lebesgue measure on \( \mathbb{R}^n \). Let \( L_\infty(I^n) \) denote the space of all real-valued, measurable, and essentially bounded functions on \( I^n \).

Our main result is the following:

THEOREM. (a) \( \forall f \in L_\infty(I^n), \)
\[
\int_{I^n} f^2 \, d\mu - \left( \int_{I^n} f \, d\mu \right)^2 \leq \left( \frac{B - b}{2} \right)^2,
\]
where \( B = \text{ess sup} \, f \) and \( b = \text{ess inf} \, f \).
(b) The estimate is the best possible, i.e. there exist \( f \in L_\infty(I^n) \) for which equality holds.

PROOF. In any Hilbert space \( \mathcal{H} \) over \( \mathbb{R} \), we have for \( u, v \in \mathcal{H}, \ |v| = 1, \)
\[
0 \leq |u|^2 - (u, v)^2 = |u - (u, v)v|^2 = \min_{\lambda \in \mathbb{R}} |u - \lambda v|^2.
\]
Now let \( \mathcal{H} = L_2(I^n) \) and \( v = 1 \). Then, for \( f \in L_2(I^n), \) (2) reduces to
\[
\|f\|^2 - (f, 1)^2 = \min_{\lambda \in \mathbb{R}} \int_{I^n} \{f(x) - \lambda\}^2 \, d\mu.
\]
For all \( f \in L_\infty(I^n), \)
\[
\left| f(x) - \frac{B + b}{2} \right| \leq \frac{B - b}{2} \quad \text{a.e.}
\]
Since \( L_\infty(I^n) \subset L_2(I^n), \) combining (3) and (4) we obtain
\[
\|f\|^2 - (f, 1)^2 \leq \int_{I^n} \left\{ f(x) - \frac{B + b}{2} \right\}^2 \, d\mu \leq \left( \frac{B - b}{2} \right)^2.
\]
This proves (a). To prove (b), for a given $B$ and $b$, and for $x \in I^n$, define $f$ as follows:

(5) \[ f(x_1, \ldots, x_n) = \begin{cases} B & \text{if } 0 \leq x_1 \leq \frac{1}{2}, \\ b & \text{if } \frac{1}{2} < x_1 \leq 1. \end{cases} \]

Then, \( \text{ess sup } f = B \), \( \text{ess inf } f = b \), and

\[ \int_{I^n} f^2 d\mu - \left( \int_{I^n} f d\mu \right)^2 = \frac{B^2 + b^2}{2} - \left( \frac{B + b}{2} \right)^2 = \left( \frac{B - b}{2} \right)^2 \]

so that equality holds in (1) for this $f \in L_\infty(I^n)$. \( \square \)

For $f \in C(I^n)$ the estimate provided in (1) becomes

(6) \[ \int_{I^n} f(x)^2 dx - \left( \int_{I^n} f(x) dx \right)^2 \leq \left( \frac{M - m}{2} \right)^2, \]

where $M$ and $m$ are the (absolute) maximum and minimum of $f$. That the estimate (6) is the best possible in $C(I^n)$ follows from part (b) of the above theorem and the fact that $C(I^n)$ is dense in $L_\infty(I^n)$. As an example, for $0 < \varepsilon < \frac{1}{2}$, consider $f_\varepsilon \in C(I^n)$ defined by

(7) \[ f_\varepsilon(x_1, \ldots, x_n) = \begin{cases} M, & \text{if } 0 \leq x_1 \leq \frac{1}{2} - \varepsilon, \\ \frac{M + m}{2} + \frac{M - m}{2} \left( \frac{\frac{1}{2} - x_1}{\varepsilon} \right), & \frac{1}{2} - \varepsilon \leq x_1 \leq \frac{1}{2} + \varepsilon, \\ m, & \frac{1}{2} + \varepsilon \leq x_1 \leq 1. \end{cases} \]

For this $f_\varepsilon(x)$ it can be shown that

(8) \[ \int_{I^n} f_\varepsilon(x)^2 dx - \left( \int_{I^n} f_\varepsilon(x) dx \right)^2 = \left( \frac{M - m}{2} \right)^2 \left( 1 - \frac{4}{3} \varepsilon \right), \]

which illustrates that the bound in (6) is the best possible in $C(I^n)$.

Finally we remark that the estimate given in (6) for the particular case $n = 1$ solves the problem posed in [1].

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REFERENCES