PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

PERFECT CONGRUENCES ON A FREE MONOID

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ABSTRACT. Perfect congruences on a free monoid X^* are characterized in terms of congruences generated by partitions of $X \cup \{1\}$. It is established that the upper semilattice of perfect congruences if $\vee$-isomorphic to the upper semilattice of partitions on $X \cup \{1\}$. A sublattice of the upper semilattice of perfect congruences is proved to be lattice isomorphic to the lattice of partitions on X.

Introduction and summary. Free monoids derive their importance from the theory of formal languages. Their homomorphic images constitute the class of all monoids, so that their congruences give rise to isomorphic copies of all monoids. In view of the richness of a free monoid, any attempt at a reasonable classification of its congruences appears to be a daunting, not to say impossible, task. The classification of a very restricted type of congruence with some information on how these fit in the lattice of all congruences may, however, be possible.

A congruence ρ on a monoid M is said to be *perfect* if the product of any two ρ-classes as *complexes* is a full ρ-class. (For related material see [**2**, VII 5.21 and VII 5.24].) In this paper we investigate the family $\mathcal{PC}(X^*)$ of perfect congruences on a free monoid X^* on the arbitrary alphabet X with three main goals in view, viz., to give an explicit description of such congruences; to determine the position of $\mathcal{PC}(X^*)$ within the lattice $\mathcal{C}(X^*)$ of all congruences on X^*; and to establish properties of the partially ordered set $\mathcal{PC}(X^*)$. We show, in particular, Theorem A, that ρ is perfect if and only if ρ is the congruence generated by the restriction of ρ to $X \cup \{1\}$. This characterization leads naturally to Theorem C wherein we prove that $\mathcal{PC}(X^*)$ is a complete $\vee$-subsemilattice of $\mathcal{C}(X^*)$ which is $\vee$-isomorphic to the complete $\vee$-semilattice $\Pi(X \cup \{1\})$ of all partitions of $X \cup \{1\}$.

Throughout the paper, for any congruence ρ on X^* and any subset T of X^*, $\rho|_T$ will denote the restriction of ρ to T and T^* will denote the monoid generated by T. If π is an equivalence relation on T, then π^* will denote the least congruence containing π and $u\pi$ the π-class of u. The difference of two sets A and B will be denoted by $A \backslash B$.

As a general reference we recommend G. Lallement's book [**1**].

We start by proving a sequence of seven lemmas thus setting the stage for the proof of Theorem A.

It will be convenient to introduce the following:

Notation. For any $A \subset X^*$ and $w \in X^*$, let w_A be the word obtained from w by deleting all letters from A which occur in w.

Received by the editors December 2, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20M05.
Key words and phrases. Perfect congruence, free monoid, $\vee$-isomorphism.
This research has been supported by Grant A4596 of the NSERC of Canada.

Now let π be a partition of $X \cup \{1\}$ and let $A = 1\pi \cap X$. Define a relation ρ_π on X^* by

$$u\rho_\pi v \Longleftrightarrow u_A(\pi|_{X\backslash A})^* v_A.$$

Clearly ρ_π is a congruence on X^*. It is, as the following lemma shows, the congruence on X^* generated by π.

LEMMA 1. *For any partition π of $X \cup \{1\}$, we have $\rho_\pi = \pi^*$.*

PROOF. Let $b, c \in X \cup \{1\}$ and suppose that $b\pi c$. If $b, c \in A$, then $b_A = 1 = c_A$, whence $b_A(\pi|_{X\backslash A})^* c_A$ and thus $b\rho_\pi c$. If $b, c \notin A$, then $b = b_A \pi|_{X\backslash A} c_A = c$, whence $b_A(\pi|_{X\backslash A})^* c_A$ and thus $b\rho_\pi c$. Therefore $\pi \subseteq \rho_\pi|_{X\backslash A}$. It follows that $\pi^* \subseteq (\rho_\pi|_{X\cup\{1\}})^* \subseteq \rho_\pi$.

Conversely, assume that $u\rho_\pi v$ for $u, v \in X^*$. Then $u_A(\pi|_{X\backslash A})^* v_A$, whence $u_A \pi^* v_A$. But clearly $u\pi^* u_A$ and $v\pi^* v_A$. Therefore $u\pi^* v$ and thus $\rho_\pi \subseteq \pi^*$.

The next two lemmas establish the sufficiency of the result: ρ is perfect if and only if ρ is the least congruence containing $\rho|_{X\cup\{1\}}$.

LEMMA 2. *Let π be a partition of X. Then π^* is perfect.*

PROOF. Let $c\pi^* ab$ so that

$$\begin{aligned}
c &= u_1 s_1 v_1 \\
u_1 t_1 v_1 &= u_2 s_2 v_2 \\
u_2 t_2 v_2 &= u_3 s_3 v_3 \\
&\vdots \\
u_{n-1} t_{n-1} v_{n-1} &= u_n s_n v_n \\
u_n t_n v_n &= ab,
\end{aligned}$$

where $s_i \pi t_i$ for all i. We wish to find $a', b' \in X^*$ such that $c = a'b', a'\pi^* a, b'\pi^* b$. The proof is by induction on the length of the above sequence. The case of length 1 is trivial. Assume the statement for length n true and suppose we have the situation as above.

By equidivisibility in X^*, the last equation above implies one of the following two cases:

1. $a = u'_n$, $b = u''_n t_n v_n$, where $u_n = u'_n u''_n$,
2. $a = u_n t_n v'_n$, $b = v''_n$, where $v_n = v'_n v''_n$.

Suppose the first case occurs; the second case is handled symmetrically. By the induction hypothesis, there exist a'', b'' such that $c = a''b'', a''\pi^* u'_n, b''\pi^* u''_n s_n v_n$. Hence $a''\pi^* a$ and $b''\pi^* u''_n s_n v_n \pi^* u''_n t_n v_n = b$ with $c = a''b''$, as required. This proves that $(a\pi^*)(b\pi^*) = (ab)\pi^*$ as sets and π^* is perfect.

LEMMA 3. *Let π be a partition of $X \cup \{1\}$. Then ρ_π $(= \pi^*)$ is perfect.*

PROOF. Let $c\rho_\pi ab$ and $A = \{x \in X | x\pi 1\}$. Then $c = u_1 v_1 u_2 v_2 \cdots u_n v_n$ for some $u_i \in A^*$, $v_i \in (X\backslash A)^*$ so that $c_A = v_1 v_2 \cdots v_n$. Further, by the definition of ρ_π, $c_A(\pi|_{X\backslash A})^* a_A b_A$. By Lemma 2, there exist $a', b' \in (X\backslash A)^*$ such that

$$c_A = a'b', \quad a'(\pi|_{X\backslash A})^* a_A, \quad b'(\pi|_{X\backslash A})^* b.$$

It follows that

$$a' = v_1 v_2 \cdots v'_i, \qquad b' = v''_i v_{i+1} \cdots v_n,$$

where $v_i = v_i' v_i''$ for some $1 \leq i \leq n$. Let

$$a'' = u_1 v_1 u_2 v_2 \cdots u_i v_i', \qquad b'' = v_i'' u_{i+1} v_{i+1} \cdots u_n v_n.$$

Then $a''_A = a'$, $b''_A = b'$, $c = a''b''$, and $a'' \rho_\pi a''_A = a' \rho_\pi a_A \rho_\pi a$ and similarly $b'' \rho_\pi b$. Therefore $(a\rho_\pi)(b\rho_\pi) = (ab)\rho_\pi$ as sets and ρ_π is perfect.

LEMMA 4. *Let ρ be a perfect congruence on X^* and $A = 1\rho \cap X$. Then $1\rho = A^*$ and for each $b \in X$, $b\rho = A^*(b\rho \cap X)A^*$.*

PROOF. We will write $\overline{u}$ for the ρ-class containing $u \in X^*$. First let $u, v \in X^*$ be such that $uv \in \overline{1}$. Then $\overline{u}\,\overline{v} = \overline{1}$ as elements of the quotient X^*/ρ and hence also as subsets of X^* since ρ is perfect. In particular, $1 = xy$ for some $x \in \overline{u}$ and $y \in \overline{v}$ whence $x = y = 1$ so that $\overline{u} = \overline{x} = \overline{1} = \overline{y} = \overline{v}$. But then $u, v \in \overline{1}$. A simple inductive argument will show that if $u_1 u_2 \cdots u_n \in \overline{1}$, then $u_1, u_2, \ldots, u_n \in \overline{1}$ for all $u_1, u_2, \ldots, u_n \in X^*$.

Since $\overline{1}$ is a submonoid of X^*, it follows that $A^* \subseteq \overline{1}$ where A^* is the (free) monoid generated by A. Conversely, if $w = x_1 x_2 \cdots x_n \in \overline{1}$, where $x_1, x_2, \ldots, x_n \in X$, then by the above, $x_1, x_2, \ldots, x_n \in \overline{1}$, that is, $x_1, x_2, \ldots, x_n \in A$ whence $w \in A^*$. Consequently $\overline{1} \subseteq A^*$ and equality prevails.

Now let $a \in X \backslash A$ and suppose that $uv \in \overline{a}$ where $u, v \in X^*$. Then, as above, $\overline{u}\,\overline{v} = \overline{a}$ as subsets of X^* whence $a = xy$ for some $x \in \overline{u}$ and $y \in \overline{v}$. Since $a \in X$, the equation $a = xy$ implies that one of x, y is equal to 1 and the other to a. It follows that one of $\overline{u}, \overline{v}$ is equal to $\overline{1}$ and the other to $\overline{a}$. A simple inductive argument will show that if $u_1 u_2 \cdots u_n \in \overline{a}$, then exactly one of $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n$ is equal to $\overline{a}$ and the remaining ones are equal to $\overline{1}$ for all $u_1, u_2, \ldots, u_n \in X^*$.

With the same a, let $B = \overline{a} \cap X$. Clearly $A^*BA^* \subseteq \overline{a}$. Conversely, if $w = x_1 x_2 \cdots x_n \in \overline{a}$ with $x_1, x_2, \ldots, x_n \in X$, then applying what we have just proved, we obtain that some $x_i \in B$ and the remaining x_j are in A and hence $w \in A^*BA^*$. Therefore $\overline{a} = A^*BA^*$.

LEMMA 5. *With the notation and hypothesis of Lemma 4, X^*/ρ is free on $(X \backslash A)\rho$.*

PROOF. We will again write $\overline{u}$ for $u\rho$. Let $a, b \in X \backslash A$ and $u, v \in X^*$ be such that $\overline{a}\,\overline{u} = \overline{b}\overline{v}$. By the formula in the preceding paragraph, we have $au = w_1 b' w_2 y$ for some $w_1, w_2 \in A^*$, $b' \in \overline{b} \cap X$, and $y \in \overline{v}$. Since $a, b \in X \backslash A$, the equation $au = w_1 b' w_2 y$ implies that $w_1 = 1$, $a = b'$, and $u = w_2 y$. Hence $\overline{a} = \overline{b}$. Clearly $w_2 y \in \overline{v}$ which, together with $\overline{u} = \overline{w_2}\overline{y}$, implies that $\overline{u} = \overline{v}$. A simple inductive argument shows that if $\overline{a}_1 \overline{a}_2 \cdots \overline{a}_s = \overline{b}_1 \overline{b}_2 \cdots \overline{b}_t$, where $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t \in X \backslash A$, then $s = t$ and $\overline{a}_i = \overline{b}_i$ for $i = 1, 2, \ldots, s$. Therefore X^*/ρ is a free monoid on the set $(X \backslash A)\rho = \{\overline{a} | a \in X \backslash A\}$.

The next lemma establishes the necessity of the result quoted before the statement of Lemma 2.

LEMMA 6. *Let ρ be a perfect congruence on X^* and let $\pi = \rho|_{X \cup \{1\}}$. Then $\rho = \pi^*$.*

PROOF. Let the notation be as in Lemma 4 and its proof. Then $A \subseteq 1\pi \subseteq 1\pi^*$ and thus $A^* \subseteq 1\pi^*$.

Any $u, v \in X^*$ can be written in the form

$$u = u_1 a_1 u_2 a_2 \cdots u_s a_s u_{s+1}, \qquad v = v_1 b_1 v_2 b_2 \cdots v_t b_t v_{t+1}$$

for some $u_i, v_i \in A^*$ and $a_i, b_i \in X\backslash A$. Assume that $u\rho v$. Then

$$\bar{a}_1\bar{a}_2\cdots\bar{a}_s = \bar{b}_1\bar{b}_2\cdots\bar{b}_t.$$

By Lemma 5, X^*/ρ is a free monoid on the set $(X\backslash A)\rho$ so that we must have $s = t$ and $\bar{a}_i = \bar{b}_i$ for $i = 1, 2, \ldots, s$. It follows that $a_i\rho b_i$ and $a_i, b_i \in X\backslash A$ which yields $a_i\pi b_i$, whence $a_i\pi^* b_i$ for $i = 1, 2, \ldots, s$. Consequently $a_1a_2\cdots a_s\pi^* b_1b_2\cdots b_s$. By the above, $A^* \subseteq 1\pi^*$ which gives

$$u_1a_1u_2a_2\cdots u_sa_su_{s+1}\pi^* v_1b_1v_2b_2\cdots v_sb_sv_{s+1},$$

that is to say $u\pi^* v$. Therefore $\rho \subseteq \pi^*$; the opposite inclusion follows by the minimality of π^*, and thus equality prevails.

The following simple result will be needed in the proofs of Theorems A and C, and is of some independent interest.

LEMMA 7. *Let π be a partition of $X \cup \{1\}$. Then $\pi^*|_{X\cup\{1\}} = \pi$.*

PROOF. For any subset T of X^*, let P_T denote the syntactic congruence of T. Recall that P_T is defined on X^* by

$$uP_Tv \text{ if } xuy \in T \iff xvy \in T \qquad (x, y \in X^*)$$

and that it is the greatest congruence on X^* saturating T. Let $A = 1\pi$, B be a π-class distinct from A, $C = A^*BA^*$, and $p, q \in X^*$.

We verify next that P_C has classes: $C, A^*, (C \cup A^*)^c$.

For $u \in C$, we have $puq \in C$ if and only if $p, q \in A^*$. It follows that C is contained in a P_C-class and thus, by saturation, C is a P_C-class.

For $u \in A^*$, we have $puq \in C$ if and only if $p \in C$ and $q \in A^*$ or $p \in A^*$ and $q \in C$. It follows that A^* is contained in a P_C-class. If $u \in A^*$, uP_Cv, and $b \in B$, then $ub \in C$ so that $vb \in C$ and hence $v \in A^*$. Thus P_C saturates A^* and therefore A^* is a P_C-class.

For $u \in X^*$, if $puq \in C$, then either $u \in A^*$ or $u \in C$. By contrapositive, $(C \cup A^*)^c$ is contained in a P_C-class. It follows from above the P_C saturates $C \cup A^*$ and so does its complement. Consequently $(C \cup A^*)^c$ is a P_C-class.

Next we show that P_{A^*} has classes: $A^*, (A^*)^c$. For $u \in A^*$, we have $puq \in A^*$ if and only if $p, q \in A^*$. Hence A^* is contained in a P_{A^*}-class and thus, by saturation, A^* is a P_{A^*}-class. If $u \notin A^*$, then $puq \notin A^*$. It follows that $(A^*)^c$ is contained in a P_{A^*}-class so, again by saturation, $(A^*)^c$ is a P_{A^*}-class.

Note that $C \cap X = B$ and $A^* \cap (X \cup \{1\}) = A$. Moreover, if B' is a π-class distinct from A and B, we also have $(C \cup A^*) \cap A^*B'A^* = \varnothing$. It now follows that the congruence $\rho = P_{A^*} \cap (\bigcap P_{A^*B'A^*})$, where B runs over all π-classes distinct from A, has the property that $\rho|_{X\cup\{1\}} = \pi$. Therefore, by minimality

$$\pi^* = (\rho|_{X\cup\{1\}})^* \subseteq \rho,$$

which implies that $\pi^*|_{X\cup\{1\}} \subseteq \rho|_{X\cup\{1\}} = \pi$. Consequently $\pi^*|_{X\cup\{1\}} \subseteq \pi$ and the opposite inclusion is obvious.

The *length* of a word w over X is the number of variables in X occurring in w, to be denoted by $\lg(w)$. An endomorphism ϕ of X^* is *length-decreasing* if for any $w \in X^*$, $\lg(w) \geq \lg(w\phi)$. Clearly, an endomorphism ϕ of X^* is length-decreasing if and only if $X\phi \subseteq X \cup \{1\}$. For any function ϕ, denote by $\overline{\phi}$ the equivalence relation induced by it.

We are now ready for the main result of the paper.

THEOREM A. *The following conditions on a congruence ρ on X^* are equivalent.*
(i) *ρ is perfect.*
(ii) *$\rho = \rho_\pi$ for some partition π of $X \cup \{1\}$.*
(iii) *$\rho = \mathcal{R}^*$ for some family $\mathcal{R}$ of relations of the form $a = b, a = 1$ where $a, b \in X$.*
(iv) *ρ is induced by a length-decreasing endomorphism of X^*.*
(v) $\rho = (\rho|_{X\cup\{1\}})^*$.

PROOF. (i) *implies* (ii). This follows directly from Lemmas 1 and 6.

(ii) *implies* (iii). Putting $a = b$ if $a\pi b$ and $a \not\pi 1$, and $a = 1$ if $a\pi 1$, we obtain a family $\mathcal{R}$ of relations. Simple reflection shows that $\pi^* = \mathcal{R}^*$ which in view of Lemma 1 yields $\rho_\pi = \mathcal{R}^*$ and thus $\rho = \mathcal{R}^*$.

(iii) *implies* (iv). Define an equivalence relation π on $X \cup \{1\}$ be specifying the π-classes as follows:

$$1\pi = \{1\} \cup \{a \in X | a = 1 \text{ is a relation in } \mathcal{R}\},$$

and for $a \in X \backslash 1\pi$,

$$a\pi = \{a\} \cup \{b \in X | a = b \text{ is a relation in } \mathcal{R}\}.$$

Let $\phi \colon X \cup \{1\} \to X \cup \{1\}$ be any mapping which induces π and for which $1\phi = 1$. Extend ϕ to an endomorphism ϕ^* of X^*. Since $X\phi^* = X\phi \subseteq X \cup \{1\}$, ϕ^* is length-decreasing. Further, since ϕ^* is an extension of ϕ, we have $\overline{\phi^*}|_{X\cup\{1\}} = \overline{\phi} = \pi$. By Lemma 7, we have $\pi^*|_{X\cup\{1\}} = \pi$ and hence by minimality, we have $\pi^* \subseteq \overline{\phi^*}$.

In order to prove the opposite inclusion, let $x, y \in X^*$ be such that $x\overline{\phi^*}y$. Then $x = x_1x_2\cdots x_m$ and $y = y_1y_2\cdots y_n$ for some $x_i, y_i \in X$ so that

$$(1) \qquad (x_1\phi)(x_2\phi)\cdots(x_m\phi) = (y_1\phi)(y_2\phi)\cdots(y_n\phi).$$

Let

$$I = \{i_1, i_2, , \ldots, i_p\} = \{i | x_i\phi \neq 1\},$$
$$J = \{j_1, j_2, \ldots, j_q\} = \{j | y_j\phi \neq 1\},$$

with $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$; this gives by (1)

$$(2) \qquad (x_{i_1}\phi)(x_{i_2}\phi)\cdots(x_{i_p}\phi) = (y_{j_1}\phi)(y_{j_2}\phi)\cdots(y_{j_q}\phi).$$

It follows that $p = q$ and $x_{i_t}\phi = y_{j_t}\phi$ for $t = 1, 2, \ldots, p$. But then $x_{i_t}\pi y_{j_t}$ and thus $x_{i_t}\pi^* y_{j_t}$ for $t = 1, 2, \ldots, p$. In addition, if $i \notin I$ and $j \notin J$, then $x_i\phi = 1 = y_j\phi$ so that $x_i\pi 1\pi y_j$, whence $x_i\pi^* 1\pi^* y_j$. Now (2) implies that

$$x = x_1x_2\cdots x_m\pi^* y_1y_2\cdots y_n = y$$

which proves that $\overline{\phi^*} \subseteq \pi^*$. Since $\rho = \pi^*$, we obtain $\rho = \pi^* = \overline{\phi^*}$.

(iv) *implies* (v). Let ρ be induced by a length-decreasing endomorphism ϕ of X^*. Let $\pi = \rho|_{X\cup\{1\}} = \overline{\phi}|_{X\cup\{1\}}$. The same argument that was used to show that $\pi^* = \overline{\phi^*}$ in the above proof shows that $\pi^* = \rho$ in the present case. Consequently $\rho = (\rho|_{X\cup\{1\}})^*$, as required.

(iv) *implies* (i). This is a direct consequence of Lemma 3.

Perfectness of the congruence induced on X^* by an endomorphism of X^* is treated in the next result.

THEOREM B. *Let $\phi\colon X^* \to X^*$ be an endomorphism. Then $\overline{\phi}$ is perfect if and only if $X\phi\backslash\{1\}$ is a code.*

PROOF. *Necessity.* Let $u_1u_2\cdots u_m = v_1v_2\cdots v_n$, where $u_i, v_i \in X\phi\backslash\{1\}$. There exist $x_i, y_i \in X$ such that $x_i\phi = u_i$ for $i = 1, 2, \ldots, m$ and $y_i\phi = v_i$ for $i = 1, 2, \ldots, n$. Hence $(x_1x_2\cdots x_m)\phi = (y_1y_2\cdots y_n)\phi$ and thus $x_1x_2\cdots x_m\overline{\phi}y_1y_2\cdots y_n$. Since $\overline{\phi}$ is perfect, by Theorem A there exists a length-decreasing endomorphism ψ of X^* such that $\overline{\phi} = \overline{\psi}$. Hence $x_1x_2\cdots x_m\overline{\psi}y_1y_2\cdots y_n$ which implies that

$$(x_1\psi)(x_2\psi)\cdots(x_m\psi) = (y_1\psi)(y_2\psi)\cdots(y_n\psi).$$

If, e.g., $x_i\psi = 1$, then $x_i\overline{\psi}1$ so $x_i\overline{\phi}1$ and thus $u_i = x_i\phi = 1$, a contradiction. Thus $x_i\psi \in X$ for $i = 1, 2, \ldots, m$ and $y_i\psi \in X$ for $i = 1, 2, \ldots, n$. The above equation then gives that $m = n$ and $x_i\psi = y_i\psi$ for $i = 1, 2, \ldots, n$. But then $x_i\phi = y_i\phi$ for $i = 1, 2, \ldots, n$ and hence $u_i = v_i$ for $i = 1, 2, \ldots, n$. Therefore $X\phi\backslash\{1\}$ is a code.

Sufficiency. By Theorem A, if suffices to show that $\overline{\phi} = (\overline{\phi}|_{X\cup\{1\}})^*$. Let $\pi = \overline{\phi}|_{X\cup\{1\}}$. Then $\pi^*|_{X\cup\{1\}} = \pi$ by Lemma 7 and thus $\pi^* \subseteq \overline{\phi}$ by minimality of π^*. For the opposite inclusion, let $x, y \in X^*$ by such that $x\overline{\phi}y$. Then $x\phi = y\phi$ and $x = x_1x_2\cdots x_m$, $y = y_1y_2\cdots y_n$ for some $x_i, y_i \in X$. Let

$$\begin{aligned} I &= \{i_1, i_2, \ldots, i_p\} = \{i | x_i\phi \neq 1\} \quad \text{with } i_1 < i_2 < \cdots < i_p, \\ J &= \{j_1, j_2, \ldots, j_q\} = \{j | y_j\phi \neq 1\} \quad \text{with } j_1 < j_2 < \cdots < j_q. \end{aligned}$$

Hence

$$x\phi = (x_1\phi)(x_2\phi)\cdots(x_m\phi) = (x_{i_1}\phi)(x_{i_2}\phi)\cdots(x_{i_p}\phi), \tag{3}$$

$$y\phi = (y_1\phi)(y_2\phi)\cdots(y_n\phi) = (y_{j_1}\phi)(y_{j_2}\phi)\cdots(y_{j_q}\phi), \tag{4}$$

where $x_{i_k}\phi, y_{j_k}\phi \in X\phi\backslash\{1\}$. The right-hand sides of (3) and (4) being equal, the hypothesis implies that $p = q$ and $x_{i_k}\phi = y_{j_k}\phi$ for $k = 1, 2, \ldots, p$. But then $x_{i_k}\pi y_{j_k}$ so also $x_{i_k}\pi^* y_{j_k}$ for $k = 1, 2, \ldots, p$. If $i \notin I$ and $j \notin J$, then $x_i\phi = 1 = x_j\phi$, whence $x_i\pi 1\pi y_j$ and thus $x_i\pi^*1\pi^*y_j$. The equality of (3) and (4) thus yields

$$x = x_1x_2\cdots x_m\pi^*y_1y_2\cdots y_n = y,$$

as required.

In the remainder of the paper, we consider the partially ordered set of perfect congruences.

THEOREM C. *Let $\Pi(X\cup\{1\})$ be the lattice of all partitions of $X\cup\{1\}$ and $\mathcal{PC}(X^*)$ be the poset of all perfect congruences on X^*, both ordered by inclusion. Then $\mathcal{PC}(X^*)$ is a complete $\vee$-sublattice of the lattice $\mathcal{C}(X^*)$ of congruences on X^* and the mapping*

$$\chi\colon \pi \to \pi^* \qquad (\pi \in \Pi(X\cup\{1\}))$$

is a complete $\vee$-isomorphism of $\Pi(X\cup\{1\})$ onto $\mathcal{PC}(X^*)$.

PROOF. Let $\{\pi_\alpha | \alpha \in I\} \subseteq \Pi(X\cup\{1\})$. For any $\beta \in I$, we have $\pi_\beta \subseteq \bigvee\pi_\alpha$, whence $\pi_\beta^* \subseteq (\bigvee\pi_\alpha)^*$ so that $\bigvee\pi_\alpha^* \subseteq (\bigvee\pi_\alpha)^*$. Conversely, $\bigvee\pi_\alpha \subseteq \bigvee\pi_\alpha^*$ implies $\bigvee\pi_\alpha \subseteq (\bigvee\pi_\alpha^*)|_{X\cup\{1\}}$, whence

$$\left(\bigvee\pi_\alpha\right)^* \subseteq \left[\left(\bigvee\pi_\alpha^*\right)\Big|_{X\cup\{1\}}\right]^* \subseteq \bigvee\pi_\alpha^*.$$

We have proved

$$\bigvee \pi_\alpha^* = \left(\bigvee \pi_\alpha\right)^*. \tag{5}$$

Now let $\{\rho_\alpha | \alpha \in I\}$ be a family of perfect congruences on X^*. Letting $\pi_\alpha = \rho_\alpha|_{X\cup\{1\}}$, by Lemma 6, we obtain $\rho_\alpha = \pi_\alpha^*$ for each $\alpha \in I$. On the other hand, Lemmas 1 and 3 yield that $(\bigvee \pi_\alpha)^*$ is perfect. Hence formula (5) implies that $\bigvee \rho_\alpha$ is perfect, that is to say, $\mathcal{PC}(X^*)$ is a complete $\vee$-sublattice of $\mathcal{C}(X^*)$. In addition, formula (5) shows that χ is a complete $\vee$-homomorphism. Injectivity of χ is a consequence of Lemma 7 while surjectivity follows from Theorem A.

PROPOSITION. *If $|X| > 1$, then the mapping χ in Theorem* C *is not a $\cap$-homomorphism.*

PROOF. Let $a, b \in X$ with $a \neq b$. Let α (respectively β) be the partition of $X \cup \{1\}$ which identifies only a (respectively b) with 1. Then for any $u, v \in X^*$, we have

$$u\rho_\alpha \cap \rho_\beta v \iff u_{\{a\}} = v_{\{a\}},\ u_{\{b\}} = v_{\{b\}}.$$

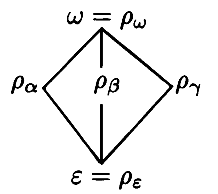
Hence $\rho_\alpha \cap \rho_\beta \neq \varepsilon$; for example $ab\rho_\alpha \cap \rho_\beta ba$. On the other hand, $\rho_{\alpha\cap\beta} = \rho_\varepsilon = \varepsilon$, the equality congruence. Therefore $\rho_\alpha \cap \rho_\beta \neq \rho_{\alpha\cap\beta}$.

We observe in passing that the above example shows that, in general, the intersection of two perfect congruences is not necessarily perfect.

EXAMPLE. Let $X = \{a, b\}$, with partitions

$$\varepsilon\colon \{a\}, \{b\}, \{1\}, \qquad \alpha\colon \{a, 1\}, \{b\},$$
$$\beta\colon \{b, 1\}, \{a\}, \quad \gamma\colon \{a, b\}, \{1\}, \quad \omega\colon \{a, b, 1\}.$$

Define θ on X^* by $u\theta v \iff u$ and v have the same number of occurrences of a and b. Then the sublattice of $\mathcal{C}(X^*)$ generated by the $\vee$-subsemilattice of perfect congruences has the following diagram:



There is a sublattice of $\mathcal{PC}(X^*)$ with the property that χ restricted to it is a lattice isomorphism. In order to treat this case, we first prove an auxiliary result.

LEMMA 8. *Let $\{\pi_\alpha | \alpha \in A\}$ be a family of partitions of X. Then $(\bigcap \pi_\alpha)^* = \bigcap \pi_\alpha^*$.*

PROOF. For any $\beta \in A$, we have $\bigcap \pi_\alpha \subseteq \pi_\beta$ and thus $(\bigcap \pi_\alpha)^* \subseteq \pi_\beta^*$, whence $(\bigcap \pi_\alpha)^* \subseteq \bigcap \pi_\alpha^*$. Conversely, let $u = a_1 a_2 \cdots a_m$ and $v = b_1 b_2 \cdots b_n$, where $a_i, b_i \in X$ and assume that $u \bigcap \pi_\alpha^*\ v$. Since $\pi_\alpha^* = \sigma_\alpha^*$, where σ_α is the partition of $X \cup \{1\}$ with $1\sigma_\alpha = 1$ and $\sigma_\alpha|_X = \pi_\alpha$, we may apply Lemma 5 which then yields that $m = n$ and $a_i \pi_\alpha^* = b_i \pi_\alpha^*$ for $i = 1, 2, \ldots, n$ and $\alpha \in A$. But then also $a_i \pi_\alpha b_i$ for all $\alpha \in A$ which gives $a_i \bigcap \pi_\alpha\ b_i$ for $i = 1, 2, \ldots, n$. Consequently $a_i (\bigcap \pi_\alpha)^* b_i$ for $i = 1, 2, \ldots, n$. It follows that $u\ (\bigcap \pi_\alpha)^*\ v$ which proves that $\bigcap \pi_\alpha^* \subseteq (\bigcap \pi_\alpha)^*$ and equality prevails.

THEOREM D. *Let $\Pi(X)$ be the lattice of all partitions of X and $\mathcal{QC}(X)$ be the poset of all perfect congruences ρ on X^* such that $1\rho = 1$, both ordered by inclusion. Then $\mathcal{QC}(X^*)$ is a complete sublattice of the lattice $\mathcal{C}(X^*)$ of congruences on X and the mapping*

$$\phi\colon \pi \to \pi^* \qquad (\pi \in \Pi(X))$$

is a complete lattice isomorphism of $\Pi(X)$ onto $\mathcal{QC}(X^*)$.

PROOF. First note that

$$\psi\colon \pi \to \pi' \qquad (\pi \in \Pi(X)),$$

where $\pi' \in \Pi(X \cup \{1\})$ with $1\pi' = 1$, $\pi'|_X = \pi$, is a complete lattice isomorphism of $\Pi(X)$ onto the sublattice of $\Pi(X \cup \{1\})$ consisting of the partitions σ such that $1\sigma = 1$. The mapping χ in Theorem B sends partitions σ of $X \cup \{1\}$ onto perfect congruences σ^* with the property $1\sigma^* = 1$ in view of Lemma 1. The same type of statement is also true about χ^{-1}. Hence Theorem B and Lemma 8 yield that $\phi = \psi\chi$ is a mapping with all required properties.

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