PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

ON THE NUMBER OF SQUARES IN A GROUP

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ABSTRACT. We show that there is a connection between the number of squares in a group and the cardinality of the group. For example, if a group has countably many squares and $x^2 = e$ implies $x = e$, then its cardinality is bounded by $2^{\aleph_0}$ and this bound can be obtained.

0. Introduction. This paper gives a negative answer to the following conjecture: If in a group without elements of order 2 there are only countably squares, then this group is countable.

In all the paper we deal with infinite groups only. In §1 we prove that if κ is an infinite cardinal and a group G without elements of order 2 has κ many squares, then the cardinality of G is bounded by 2^κ. The proof is technical and uses the Erdős-Rado theorem. In §2 we construct an example of a torsion-free group G of power continuum with only countably many squares, which according to the results from §1 is the best possible. The construction easily generalizes (Corollary 2.5) to the case of λ-many squares where λ is an arbitrary infinite cardinal; however in this case we are able to construct a group whose cardinality is that of a linear ordering with a dense subset of cardinality λ.

One could ask if the above results easily generalize when squares are replaced by higher powers. For $k > 1$ let $G^k = \{x^k : x \in G\}$. When dealing with kth powers, the natural assumption on a group is that $x^k = e$ implies $x = e$. After the first version of this paper was written, Professor G. Higman pointed out to the author the following:

(1) If we want to prove for some f satisfying $f(\kappa) \geq 2^\kappa$ that, for any G, $|G| \leq f(|G^k|)$, then w.l.o.g. we may assume that G^k is contained in the center of G.

(2) If Burnside's group of exponent k with 2 generators is finite, then for any G satisfying $x^k = e \to x = e$ we have $|G| \leq 2^{|G^k|}$.

Thus (2) provides us with another, maybe more algebraic, proof of Theorem 1.3, and gives us an estimation for G in case $k = 2, 3, 4, 6$. However, (1) enabled the author to strengthen his previous estimations and prove that $|G| \leq 2^{|G^k|}$ for $k = 2^n 3^m$ where $n \geq 0$ and $m = 0$ or 1. For other k's the question of whether any estimation exists remains at present open. However, at least we cannot prove that for some $k > 1$, for any G, $|G| = |G^k|$, as the construction from §2 generalizes to the case of kth powers for any $k > 2$. In contrast to the situation with infinite groups, we trivially have that if G^k is finite and G satisfies $x^k = e \to x = e$, then $G = G^k$. Notice that constructing for some k a group G such that $|G| > 2^{|G^k|}$ (and $x^k = e \to x = e$ holds) would enable us to prove that Burnside's group of exponent k is infinite.

Received by the editors January 31, 1985 and, in revised form, May 1, 1985 and November 1, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E34; Secondary 20F05.

We use the standard set-theoretical notation. So, for example, ${}^{<\omega}X$ denotes the set of all finite sequences of elements of X. For a finite sequence $\overrightarrow{\eta}$, $\iota(\overrightarrow{\eta})$ denotes the length of $\overrightarrow{\eta}$.

1. Negative results. Throughout this section we assume that G is a group without elements of order 2.

FACT 1.1 (A SPECIAL CASE OF THE ERDŐS-RADO THEOREM, SEE [**1**, **3**]). Let μ be an infinite cardinal. Assume that f is a binary function on $(2^\mu)^+$ such that for every $\alpha < \beta < (2^\mu)^+$ we have $f(\alpha,\beta) \in \mu$. Then there exists $\delta < \mu$ and $A \subseteq (2^\mu)^+$, $|A| = \mu^+$, such that, for every $\alpha, \beta \in A$, if $\alpha < \beta$, then $f(\alpha,\beta) = \delta$.

LEMMA 1.2. *Assume that* $a, b, c, d \in G$, $(ab)^2 = (cd)^2$, $(bc)^2 = (bd)^2$, *and* $a^2 = b^2 = c^2 = d^2$. *Then* $a = b$ *and* $c = d$.

PROOF. We have $bcbc = bdbd$, so $cbc = dbd$. Multiplying this equation on the right side by d and on the left by c we obtain $c^2bcd = cdbd^2$. From the assumptions it follows that c^2 commutes with b, c, d, and equals d^2, so we get $bcd = cdb$. Now $(ab)^2b = (cd)^2b = b(cd)^2 = b(ab)^2$. So $(ab)^2b = b(ab)^2 = babab = (ba)^2b$, hence $(ab)^2 = (ba)^2$. From this we get

$$(ab)^4 = (ab)^2(ba)^2 = ababbaba = b^2b^2a^2a^2 = a^8.$$

Hence $(a^{-2}ab)^4 = e$. But in G there are no elements of order 2, so $a^{-2}ab = a^{-1}b = e$. We conclude that $a = b$. Repeating the above proof for c, d, a, b instead of a, b, c, d, we see that $c = d$. □

THEOREM 1.3. *Assume that G is an infinite group. Then* $|G| \leq 2^{|G^2|}$.

PROOF. By the remark in the Introdution we can assume that G^2 is infinite. Let $|G^2| = \mu$. Suppose that $|G| > 2^\mu$. Then we can find in G pairwise distinct elements $\{a_\alpha : \alpha < (2^\mu)^+\}$ such that $a_\alpha^2 = a_\beta^2$ for every $\alpha < \beta < (2^\mu)^+$. For $\alpha < \beta < (2^\mu)^+$ let $f(\alpha,\beta) = (a_\alpha a_\beta)^2 \in G^2$. From Fact 1.1 it follows that we can choose an infinite set $A \subseteq (2^\mu)^+$ such that for $\alpha, \beta, \alpha', \beta' \in A$ if $\alpha < \beta$, $\alpha' < \beta'$ then $f(\alpha,\beta) = f(\alpha',\beta')$. W.l.o.g. we can assume that $\{a_n : n < \omega\} \subseteq A$. So we have $a_0^2 = a_1^2 = a_2^2 = a_3^2$, $(a_0a_1)^2 = (a_2a_3)^2$, and $(a_1a_2)^2 = (a_1a_3)^2$. From Lemma 1.2 we conclude that $a_0 = a_1$ and $a_2 = a_3$, a contradiction. □

DEFINITION 1.4. For a given complete theory T in a first-order language L, a formula $\varphi(\overline{x};\overline{y})$ of L is stable in T if for every model M of T there are $\leq \|M\|$ φ-types over M. If φ is not stable, we call it unstable.

For basic results on stability, see, for example, [**2**, or **3**]. If G is a group, then we say that $\varphi(\overline{x};\overline{y})$ is stable in G instead of "stable in $T = \mathrm{Th}(G)$."

THEOREM 1.5. *Assume that G is an infinite group. If* $\varphi(x;y) \equiv xy = yx$ *is stable in G, then* $|G| = |G^2|$.

PROOF. If not, let $|G^2| = \mu$, and choose $\{a_\alpha : \alpha < \mu^+\} \subseteq G$ such that $a_\alpha^2 = a_\beta^2$ for $\alpha < \beta < \mu^+$ ($\mu \geq \aleph_0$ by Theorem 1.3).

By induction one can easily construct three sequences $\{b_n\}_{n<\omega}$, $\{c_n\}_{n<\omega} \subseteq G$ and $\{B_n\}_{n<\omega}$, $B_n \subseteq G$ for every n, such that

(1) $b_0 = a_0$,

(2) $b_n \in B_n \backslash B_{n+1}$, $B_{n+1} \subseteq B_n \subseteq \{a_\alpha : \alpha < \mu^+\}$ for $n < \omega$,

(3) $|B_n| = \mu^+$,

(4) if $c, d \in B_{n+1}$ then $(b_n c)^2 = (b_n d)^2 = c_n \in G^2$.

Now if there are $n < m$ such that $c_{2n} = c_{2m}$, then taking $b_{2n}, b_{2n+1}, b_{2m}, b_{2m+1}$ we see that

$$(b_{2n}b_{2n+1})^2 = (b_{2m}b_{2m+1})^2, \qquad (b_{2n+1}b_{2m+1})^2 = (b_{2n+1}b_{2m})^2$$

and

$$b_{2n}^2 = b_{2n+1}^2 = b_{2m}^2 = b_{2m+1}^2.$$

So Lemma 1.2 leads to a contradiction. Thus for every $n < m$, $c_{2n} \neq c_{2m}$. From this we get that, for $n \neq m$, $(b_{4n}b_{4m})^2 = (b_{4n}b_{4m+2})^2$ holds if and only if $n < m$. But $(b_{4n}b_{4m})^2 = (b_{4n}b_{4m+2})^2$ is equivalent to $b_{4n}b_{4m}b_{4m+2} = b_{4m}b_{4m+2}b_{4n}$, so to $\varphi(b_{4n}; b_{4m}b_{4m+2})$. It follows that using $\varphi(x;y)$ we can define on some infinite subset of G a linear order. From the facts proved in [**3**] we can easily deduce that $\varphi(x;y)$ is unstable in G. □

2. An example. In this section we shall construct an example of a group G of power continuum, without elements of order 2, with only countably many squares. From Theorem 1.3 it follows that such an example is the best possible for a countable G^2, i.e. $|G|$ is maximal possible.

The construction consists of defining a suitable normal subgroup H_0 of a free group G_0 of power continuum. G will be the quotient group G_0/H_0. Let G_0 be the free group generated by the set of letters $\{\eta \colon \eta \in {}^{\omega}2\}$. We identify finite sequences of elements of ${}^{\omega}2$ (i.e. elements of ${}^{<\omega}({}^{\omega}2)$) with appropriate words—element of G_0. Recall that $k = \{i \colon i < k\}$ for $k \in \omega$.

DEFINITION 2.1. For every $\vec{\eta} \in {}^{<\omega}({}^{\omega}2)$ we define $\vec{h}(\vec{\eta}) \in {}^{<\omega}({}^{<\omega}2)$ as follows: Let $\iota(\vec{\eta}) = n$. For $i < n$ let

$$k_i = \max_{i \neq j < n} \min\{t+1 : t < \omega \ \&\ \vec{\eta}[i] \restriction t \neq \vec{\eta}[j] \restriction t\}.$$

Here $\min\varnothing = \max\varnothing = 0$. Define $\vec{h}(\vec{\eta})[i] = \vec{\eta}[i] \restriction k_i$ for every $i < n$, and $\iota(\vec{h}(\vec{\eta})) = n$.

We define now a subgroup H_0 of G_0 as follows:

DEFINITION 2.2. (1) $R = \{(\vec{\eta})^2(\vec{\nu})^{-2} \colon \vec{\eta}, \vec{\nu} \in {}^{<\omega}({}^{\omega}2) \ \&\ \vec{h}(\vec{\eta}) = \vec{h}(\vec{\nu})\}$.

(2) Let H_0 be the normal subgroup of G_0 generated by R, and let $G = G_0/H_0$.

For notational simplicity we treat words in G_0 as names for their images in G under the canonical projection.

THEOREM 2.3. *$|G| = 2^{\aleph_0}$, $|G^2| = \aleph_0$ and in G there are no elements of order* 2.

We prove Theorem 2.3 in a series of lemmas.

LEMMA 1. *If $\eta, \nu \in {}^{\omega}2$, $\eta \neq \nu$ then, in G, $\eta \neq \nu$.*

PROOF. Consider a homomorphism $f \colon G_0 \to \mathbf{Z}_2$ such that $f(\eta) = 0$ and $f(\nu) = 1$. Then $H_0 \subseteq \operatorname{Ker} f$. $0 \neq 1$, so, in $G = G_0/H_0$, $\eta \neq \nu$. □

Notice that if $\eta, \nu \in {}^{\omega}2$, $\eta \neq \nu$, then $\vec{h}(\langle\eta\rangle) = \vec{h}(\langle\nu\rangle) = \langle\varnothing\rangle$, so in G we have $\eta^2 = \nu^2$. Denote this common square by α, i.e. for all $\eta \in {}^{\omega}2$, $\eta^2 = \alpha$ in G.

LEMMA 2. *Every element of G is equal to $\alpha^k\vec{\eta}$ for some $k \in \mathbf{Z}$ and $\vec{\eta} \in {}^{<\omega}({}^{\omega}2)$ such that, for $0 \le i < \iota(\vec{\eta}) - 1$, $\vec{\eta}[i] \ne \vec{\eta}[i+1]$.*

PROOF. Notice that for $\eta \in {}^{\omega}2$, in G, $\eta^{-1} = \alpha^{-1}\eta$; $\eta\eta = \alpha$; and α, α^{-1} commute with ν, ν^{-1} for every $\nu \in {}^{\omega}2$. □

LEMMA 3. $|G^2| \le \aleph_0$.

PROOF. The value of $(\alpha^k\vec{\eta})^2 = \alpha^{2k}(\vec{\eta})^2$ depends only on k and $\vec{h}(\vec{\eta})$, so there are only countably many possibilities. □

LEMMA 4. *In G: If $(\alpha^k\vec{\eta})^2 = e$, then $\alpha^k\vec{\eta} = e$.*

PROOF. We prove it by induction on $|\mathrm{Rng}(\vec{\eta})|$. We may assume that $\vec{\eta}$ is such as in Lemma 2.

(a) If $|\mathrm{Rng}(\vec{\eta})| = 0$ then $\vec{\eta} = \varnothing$, and we have $(\alpha^k)^2 = \alpha^{2k} = e$.

(b) If $|\mathrm{Rng}(\vec{\eta})| = 1$ then $\vec{\eta} = \langle\eta\rangle$ for some $\eta \in {}^{\omega}2$, so we have

$$(\alpha^k\vec{\eta})^2 = \alpha^{2k}\eta\eta = \alpha^{2k+1}.$$

Consider now the homomorphism $f: G_0 \to (\mathbf{Z}, +)$, such that, for every $\eta \in {}^{\omega}2$, $f(\eta) = 1$ in $\mathbf{Z}$. Then clearly $H_0 \subseteq \operatorname{Ker} f$, so whenever $f(x) \ne f(y)$ in $\mathbf{Z}$, $x \ne y$ holds in G. (For simplicity we do not distinguish explicitly between f and the induced homomorphism $\bar{f}: G \to (\mathbf{Z}, +)$.) So in case (a) we get $f(\alpha^{2k}) = 4k = 0$, hence $k = 0$. Similarly, in case (b) we get $4k + 2 = 0$, a contradiction.

(c) Assume that $|\mathrm{Rng}(\vec{\eta})| = n \ge 2$, and for all $\alpha^{\iota}\vec{\nu}$ such that $|\mathrm{Rng}(\vec{\nu})| < n$, Lemma 4 holds. Assume that $\alpha^k\vec{\eta})^2 = e$. We have to prove that $\alpha^k\vec{\eta} = e$.

Let us choose two letters $\eta, \nu \in \mathrm{Rng}(\vec{\eta})$ such that

$$\iota(\vec{h}(\langle\eta, \nu\rangle)[0]) = \iota(\vec{h}(\langle\eta, \nu\rangle)[1])$$

is maximal possible. If there are two or more such pairs $\eta, \nu \in \mathrm{Rng}(\vec{\eta})$, we choose one of them. We prove that there is an inner automorphism of G which maps $\alpha^k\vec{\eta}$ to some $\alpha^{\iota}\vec{\eta}_1$ such that $(\alpha^{\iota}\vec{\eta}_1)^2 = e$ and $|\mathrm{Rng}(\vec{\eta}_1)| < n$. Clearly by the induction hypothesis this will finish the proof of Lemma 4.

CLAIM A. If $\vec{\eta} = \vec{\eta}_0{}^\frown\langle\eta\rangle{}^\frown\vec{\eta}_1{}^\frown\langle\eta\rangle{}^\frown\vec{\eta}_2$, and $\eta, \nu \notin \mathrm{Rng}(\vec{\eta}_1)$, then, in G, $\eta\vec{\eta}_1\eta = \nu\vec{\eta}_1\nu$.

PROOF OF CLAIM A. Notice that from the maximality of the choice of η, ν and from the definition of $\vec{h}$ it follows that

$$\vec{h}(\vec{\eta}_1{}^\frown\langle\eta\rangle) = \vec{h}(\vec{\eta}_1{}^\frown\langle\nu\rangle).$$

So in G, $\vec{\eta}_1\eta\vec{\eta}_1\eta = \vec{\eta}_1\nu\vec{\eta}_1\nu$. Multiplying on the left side by $(\vec{\eta}_1)^{-1}$ we see that Claim A holds.

From Claim A follows at once

CLAIM B. If $\vec{\eta} = \vec{\eta}_0{}^\frown\langle\eta\rangle{}^\frown\vec{\eta}_1{}^\frown\langle\nu\rangle{}^\frown\vec{\eta}_2{}^\frown\langle\nu\rangle{}^\frown\vec{\eta}_3$ and $\eta, \nu \notin \mathrm{Rng}(\vec{\eta}_1{}^\frown\vec{\eta}_2)$ then, in G, $\vec{\eta} = \vec{\eta}_0{}^\frown\langle\nu\rangle{}^\frown\vec{\eta}_1{}^\frown\langle\nu\rangle{}^\frown\vec{\eta}_2{}^\frown\langle\eta\rangle{}^\frown\vec{\eta}_3$.

PROOF. Apply Claim A twice to get

$$\eta\vec{\eta}_1\nu\vec{\eta}_2\nu = \eta\vec{\eta}_1\eta\vec{\eta}_2\eta = \nu\vec{\eta}_1\nu\vec{\eta}_2\eta.$$

From Claim B it follows that $\overrightarrow{\eta}$ equals in G some other word $\overrightarrow{\eta}'$ such that if we delete in $\overrightarrow{\eta}'$ the fragments not containing ν or η, then we obtain a word of the form

(1) $\eta^{2\iota}\overrightarrow{\nu}$ or

(2) $\nu^{2\iota}\overrightarrow{\nu}$, where $\overrightarrow{\nu} = \langle \nu, \eta, \nu, \eta, \dots \rangle$ or $\overrightarrow{\nu} = \langle \eta, \nu, \eta, \nu, \dots \rangle$ and $\iota \in \mathbf{Z}$, $\iota \geq 0$.

By Claim A we can replace $\eta^{2\iota}$ by $\nu^{2\iota}$. There are two cases.

CASE I. $\iota(\overrightarrow{\nu})$ is odd.

In this case the first and the last member of $\overrightarrow{\nu}$ are the same. Assume w.l.o.g. that $\overrightarrow{\nu} = \langle \nu \rangle ^\frown \overrightarrow{\nu}' {}^\frown \langle \nu \rangle$ and (2) holds. Let $\overrightarrow{\eta}' = \overrightarrow{\eta}_0 {}^\frown \langle \nu \rangle ^\frown \overrightarrow{\eta}_1 {}^\frown \langle \nu \rangle ^\frown \overrightarrow{\eta}_2$ where $\nu, \eta \notin \mathrm{Rng}(\overrightarrow{\eta}_0 {}^\frown \overrightarrow{\eta}_2)$. Then the inner automorphism φ defined by

$$\varphi(x) = \nu \overrightarrow{\eta}_2 x (\nu \overrightarrow{\eta}_2)^{-1}$$

maps $\alpha^k \overrightarrow{\eta}'$ onto $\alpha^k \nu \overrightarrow{\eta}_2 \overrightarrow{\eta}_0 \nu \overrightarrow{\eta}_1 = \alpha^k \overrightarrow{\eta}''$.

Now when we delete in $\overrightarrow{\eta}''$ all letters distinct from ν, η, we obtain the word $\nu^{2\iota+2}\overrightarrow{\nu}'$, and $\iota(\overrightarrow{\nu}') = \iota(\overrightarrow{\nu}) - 2$. Hence, iterating this process finitely many times, we get the word $\alpha^{\iota}\overrightarrow{\eta}_1$ (after possible reductions according to Lemma 2) such that $\mathrm{Rng}(\overrightarrow{\eta}_1) \subseteq \mathrm{Rng}(\overrightarrow{\eta})$ and $\eta \notin \mathrm{Rng}(\overrightarrow{\eta}_1)$ or $\nu \notin \mathrm{Rng}(\overrightarrow{\eta}_1)$, so in this case the induction step is done.

CASE II. $\iota(\overrightarrow{\nu})$ is even. If $\overrightarrow{\nu} = \emptyset$, we finish, for then $\eta \notin \mathrm{Rng}(\overrightarrow{\eta}')$ or $\nu \notin \mathrm{Rng}(\overrightarrow{\eta}')$ and $\mathrm{Rng}(\overrightarrow{\eta}') \subseteq \mathrm{Rng}(\overrightarrow{\eta})$. So assume that $\overrightarrow{\nu} \neq \emptyset$.

Let us consider now the free group G_1 generated by free generators a and b. Let H_1 be the normal subgroup of G_1 generated by the set $\{a^2, b^2\}$, and let $G_2 = G_1/H_1$. The elements of G_2 can be written down in a very simple form, namely as finite sequences of the form $ababab\cdots$ or $bababa\cdots$ (the length clearly may be even or odd), and any two such sequences are equal in G_2 iff they are equal.

We define some homomorphism $f: G_0 \to G_2$ (it suffices to define $F(\mu)$ for every $\mu \in {}^{\omega}2$).

(i) If $h(\langle \eta, \nu \rangle)[0] \subseteq \mu$, then let $f(\mu) = a$.

(ii) If $h(\langle \eta, \nu \rangle)[1] \subseteq \mu$, then let $f(\mu) = b$.

(iii) If neither (i) nor (ii), then let $f(\mu) = e$.

Now, from the definitions of h and H_0 it follows that $H_0 \subseteq \operatorname{Ker} f$. From the choice of η, ν it follows that $f(\alpha^k \overrightarrow{\eta}')$ is equal to $f(\overrightarrow{\nu})$, and $f(\overrightarrow{\nu})$ equals $(ab)^{\iota}$ or $(ba)^{\iota}$ for some $\iota > 0$.

But we know that, in G, $(\alpha^n \overrightarrow{\eta}')^2 = e$, so, in G_2, $[(ab)^{\iota}]^2 = (ab)^{2\iota} = e$, a contradiction, because, in G_2, $(ab)^{2\iota} = e$ only when $\iota = 0$. □

LEMMA 5. $|G^2| \geq \aleph_0$.

PROOF. This follows immediately from Theorem 1.3, but we can give also another, more direct proof. For $\eta_0, \eta_1 \in {}^{\omega}2$, $\eta_0 \neq \eta_1$ let $g(\eta_0, \eta_1)$ be $\eta_o \restriction \min\{k\colon \eta_0[k] \neq \eta_1[k]\}$. Now if $\eta_0 \neq \eta_1$, $\nu_0 \neq \nu_1$, $\eta_i, \nu_i \in {}^{\omega}2$ and $g(\eta_0, \eta_1) \neq g(\nu_0, \nu_1)$, then by choosing a suitable homomorphism $f: G_0 \to G_2$ with $f(\eta_0\eta_1)^2$, $f(\nu_0\nu_1)^2$ distinct (namely one of them equal to e and the other distinct from e in G_2), we may prove as in Lemma 4 that, in G, $(\eta_0\eta_1)^2 \neq (\nu_0\nu_1)^2$. □

Clearly the series of Lemmas 1–5 proves Theorem 2.3. □

REMARK 1. A similar argument shows that G is torsion-free.

REMARK 2. In the proof of Lemma 4 we have in fact found an algorithm deciding whether $\alpha^n \overrightarrow{\eta} = e$ in G or not. Define

$$h_{ij}(\overrightarrow{\eta}) = \sup\{n : \overrightarrow{\eta}[i] \upharpoonright n = \overrightarrow{\eta}[j] \upharpoonright n\} \quad \text{for } i, j < \iota(\overrightarrow{\eta})$$

(with $\sup \omega = \omega$). We see from the proof of Lemma 4 that if we have $\overrightarrow{\nu}$, $\overrightarrow{\eta}$ of the same length and, for every $i, j, i', j' < \iota(\overrightarrow{\eta}) = \iota(\overrightarrow{\nu})$,

$$h_{ij}(\overrightarrow{\nu}) < h_{i'j'}(\overrightarrow{\nu}) \quad \text{iff} \quad h_{ij}(\overrightarrow{\eta}) < h_{i'j'}(\overrightarrow{\eta}),$$

and

$$h_{ij}(\overrightarrow{\nu}) < \omega \quad \text{iff} \quad h_{ij}(\overrightarrow{\eta}) < \omega,$$

then, in G, $\alpha^n \overrightarrow{\eta} = e$ iff $\alpha^n \overrightarrow{\nu} = e$ (it follows from the form of the algorithm mentioned above).

DEFINITION 2.4 (SEE [**2** OR **3**]). For an infinite cardinal λ define $\operatorname{Ded} \lambda$ as the first cardinal μ such that there is no dense linear order of cardinality μ with a dense subset of cardinality λ.

Notice that $\lambda^+ < \operatorname{Ded} \lambda \leq (2^\lambda)^+$, $\operatorname{cf}(\operatorname{Ded} \lambda) > \lambda$.

COROLLARY 2.5. *Let $\aleph_0 \leq \lambda \leq \mu < \operatorname{Ded} \lambda$. Then there exists a torsion-free group G of cardinality μ such that $|G^2| = \lambda$.*

PROOF. Consider the tree ${}^{\leq\lambda}2$. Then we can choose a subtree of ${}^{\leq\lambda}2$ such that it has μ branches of length λ, and when we generate on the set of these branches a group G (as in Definitions 2.1 and 2.2) then the set G^2 has cardinality λ. □

CONJECTURE. If G is infinite and has no elements of order 2, then $|G| < \operatorname{Ded}(|G^2|)$.

Changing somewhat Definition 2.2 one can construct analogously for any $k > 2$ a torsion-free group G of power continuum with $|G^k| = \aleph_0$. (In the proof of Lemma 4 one should use instead of G_2 the group $G_k = G_1/H_k$, where H_k is the normal subgroup of G_1 generated by $\{a^k, b^k\}$.) The counterpart of Corollary 2.5 holds as well.

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