

COVERING ÉTENDUES AND FREYD'S THEOREM

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ABSTRACT. From Freyd's covering theorem for Grothendieck topoi, it immediately follows that every Grothendieck topos \mathcal{E} admits a hyperconnected geometric morphism $\mathcal{F} \rightarrow \mathcal{E}$, where \mathcal{F} is an étendue of (discrete) G -sheaves. As a corollary, we obtain that \mathcal{E} admits an open surjection from a localic topos.

A fundamental representation theorem in topos theory is provided by the following result of P. Freyd. If \mathcal{E} is a Grothendieck topos, there is a connected atomic geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, where \mathcal{F} is localic over a topos $\mathcal{C}(G)$ of continuous G -sets, G being a topological group. In fact, G may be chosen to be either G_0 , the group of permutations of \mathbb{N} , or G_1 , the group of order-preserving permutations of \mathbb{Q} . (See [1] or the excellent survey article [2].) In [4], it was shown that given \mathcal{E} , there is a hyperconnected geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, where \mathcal{F} is an étendue, i.e., there is a surjective local homeomorphism $\mathcal{F}/X \rightarrow \mathcal{F}$ with \mathcal{F}/X localic. Using Freyd's result, we not only obtain this theorem directly (with \mathcal{F} a particularly nice étendue), but the fact that \mathcal{E} has a localic cover via an open surjection [3, p. 61] also follows. To fix some notation, $\text{sh}(G; B)$ denotes the topos of (discrete) G -sheaves, where B is a locale on which the group G acts. $\text{sh}(G; B)$ is a basic example of étendue and is locally $\text{sh}(B)$. Also, S^G denotes the topos of G -sets. There is a canonical map $S^G \xrightarrow{g} \mathcal{C}(G)$ [4].

THEOREM. *Let \mathcal{E} be a Grothendieck topos. There is a hyperconnected geometric morphism $\mathcal{F} \xrightarrow{h} \mathcal{E}$, where \mathcal{F} is an étendue of G -sheaves, $\text{sh}(G; B)$. (G can be taken to be either the group G_0 or G_1 .)*

PROOF. Let $\mathcal{C}(G)[A] \xrightarrow{f} \mathcal{E}$ be connected and atomic, where $\mathcal{C}(G)[A]$ is the topos of $\mathcal{C}(G)$ -sheaves on an internal locale A in $\mathcal{C}(G)$. Let B be the Macneille completion of A . Thus A is the cocontinuous reflection of B along the hyperconnected geometric morphism $S^G \xrightarrow{g} \mathcal{C}(G)$. The following diagram can be seen to be a pullback.

$$\begin{array}{ccc} \text{sh}(G; B) & \xrightarrow{\bar{g}} & \mathcal{C}(G)[A] \\ \downarrow \bar{k} & & \downarrow k \\ S^G & \xrightarrow{g} & \mathcal{C}(G) \end{array}$$

Since pullbacks perserve hyperconnectedness [3, p. 54], \bar{g} is hyperconnected. Connected and atomic implies hyperconnected [2, Lemma 2.1], thus the composite $\text{sh}(G; B) \xrightarrow{\bar{g}} \mathcal{C}(G)[A] \xrightarrow{f} \mathcal{E}$ is hyperconnected. Take this to be h . \square

Received by the editors January 31, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 18B25.

Key words and phrases. Grothendieck topos, étendue, localic topos.

REMARK. $\mathrm{sh}(G; B) \xrightarrow{\bar{k}} \mathcal{S}^G$ is localic and \mathcal{S}^G is Boolean, thus our étendue satisfies that every object is a quotient of decidable object [2, Proposition 4.3].

COROLLARY. *Let \mathcal{E} be a Grothendieck topos. There is an open surjection $\mathrm{sh}(B) \rightarrow \mathcal{E}$, where B is a locale on which G_0 (or G_1) acts.*

PROOF. Compose the hyperconnected map $\mathrm{sh}(G; B) \xrightarrow{h} \mathcal{E}$ with the surjective local homeomorphism $\mathrm{sh}(B) \rightarrow \mathrm{sh}(G; B)$. \square

Using this open surjection in the Joyal-Tierney representation theorem [3, Theorem VIII, 3.2], we have \mathcal{E} is equivalent to a topos of Γ -sheaves, where Γ is a localic groupoid with locale of objects B (on which $G_0(G_1)$ acts).

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