COVERING ÉTENDUES AND FREYD'S THEOREM

KIMMO I. ROSENTHAL

ABSTRACT. From Freyd's covering theorem for Grothendieck topoi, it immediately follows that every Grothendieck topos \( \mathcal{E} \) admits a hyperconnected geometric morphism \( \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F} \) is an étendue of (discrete) \( G \)-sheaves. As a corollary, we obtain that \( \mathcal{E} \) admits an open surjection from a localic topos.

A fundamental representation theorem in topos theory is provided by the following result of P. Freyd. If \( \mathcal{E} \) is a Grothendieck topos, there is a connected atomic geometric morphism \( \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F} \) is localic over a topos \( C(G) \) of continuous \( G \)-sets, \( G \) being a topological group. In fact, \( G \) may be chosen to be either \( G_0 \), the group of permutations of \( \mathbb{N} \), or \( G_1 \), the group of order-preserving permutations of \( \mathbb{Q} \). (See [1] or the excellent survey article [2].) In [4], it was shown that given \( \mathcal{E} \), there is a hyperconnected geometric morphism \( \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F} \) is an étendue, i.e., there is a surjective local homeomorphism \( \mathcal{F}/X \to \mathcal{F} \) with \( \mathcal{F}/X \) localic. Using Freyd's result, we not only obtain this theorem directly (with \( \mathcal{F} \) a particularly nice étendue), but the fact that \( \mathcal{E} \) has a localic cover via an open surjection [3, p. 61] also follows. To fix some notation, \( \text{sh}(G; B) \) denotes the topos of (discrete) \( G \)-sheaves, where \( B \) is a locale on which the group \( G \) acts. \( \text{sh}(G; B) \) is a basic example of étendue and is locally \( \text{sh}(B) \). Also, \( S^G \) denotes the topos of \( G \)-sets. There is a canonical map \( S^G \to C(G) \) [4].

THEOREM. Let \( \mathcal{E} \) be a Grothendieck topos. There is a hyperconnected geometric morphism \( \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F} \) is an étendue of \( G \)-sheaves, \( \text{sh}(G; B) \). (\( G \) can be taken to be either the group \( G_0 \) or \( G_1 \).)

PROOF. Let \( C(G)[A] \to \mathcal{E} \) be connected and atomic, where \( C(G)[A] \) is the topos of \( C(G) \)-sheaves on an internal locale \( A \) in \( C(G) \). Let \( B \) be the Macneille completion of \( A \). Thus \( A \) is the cocontinuous reflection of \( B \) along the hyperconnected geometric morphism \( S^G \to C(G) \). The following diagram can be seen to be a pullback.

\[
\begin{array}{ccc}
\text{sh}(G; B) & \to & C(G)[A] \\
\downarrow \bar{g} & & \downarrow k \\
S^G & \to & C(G)
\end{array}
\]

Since pullbacks preserve hyperconnectedness [3, p. 54], \( \bar{g} \) is hyperconnected. Connected and atomic implies hyperconnected [2, Lemma 2.1], thus the composite \( \text{sh}(G; B) \to C(G)[A] \to \mathcal{E} \) is hyperconnected. Take this to be \( h \). \( \square \)

Received by the editors January 31, 1986.


Key words and phrases. Grothendieck topos, étendue, localic topos.

\(^{\text{©1987 American Mathematical Society}}\)

\(^{0002-9939/87 \$1.00 + \$0.25 per page}\)

221
REMARK. \( \text{sh}(G; B) \xrightarrow{\mathcal{E}} S^G \) is localic and \( S^G \) is Boolean, thus our étendue satisfies that every object is a quotient of decidable object [2, Proposition 4.3].

COROLLARY. Let \( \mathcal{E} \) be a Grothendieck topos. There is an open surjection \( \text{sh}(B) \to \mathcal{E} \), where \( B \) is a locale on which \( G_0 \) (or \( G_1 \)) acts.

PROOF. Compose the hyperconnected map \( \text{sh}(G; B) \xrightarrow{h} \mathcal{E} \) with the surjective local homeomorphism \( \text{sh}(B) \to \text{sh}(G; B) \). □

Using this open surjection in the Joyal-Tierney representation theorem [3, Theorem VIII, 3.2], we have \( \mathcal{E} \) is equivalent to a topos of \( \Gamma \)-sheaves, where \( \Gamma \) is a localic groupoid with locale of objects \( B \) (on which \( G_0(G_1) \) acts).

REFERENCES

1. P. Freyd, All topoi are localic, or Why permutation models prevail (unpublished typescript, Univ. of Pennsylvania, 1979).

DEPARTMENT OF MATHEMATICS, UNION COLLEGE, SCHENECTADY, NEW YORK 12308