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# COVERING ÉTENDUES AND FREYD'S THEOREM

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ABSTRACT. From Freyd's covering theorem for Grothendieck topoi, it immediately follows that every Grothendieck topos $\mathcal{E}$ admits a hyperconnected geometric morphism $\mathcal{F} \to \mathcal{E}$, where $\mathcal{F}$ is an étendue of (discrete) $G$-sheaves. As a corollary, we obtain that $\mathcal{E}$ admits an open surjection from a localic topos.

A fundamental representation theorem in topos theory is provided by the following result of P. Freyd. If $\mathcal{E}$ is a Grothendieck topos, there is a connected atomic geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, where $\mathcal{F}$ is localic over a topos $\mathcal{C}(G)$ of continuous $G$-sets, $G$ being a topological group. In fact, $G$ may be chosen to be either $G_0$, the group of permutations of $\mathbf{N}$, or $G_1$, the group of order-preserving permutations of $\mathbf{Q}$. (See [**1**] or the excellent survey article [**2**].) In [**4**], it was shown that given $\mathcal{E}$, there is a hyperconnected geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, where $\mathcal{F}$ is an étendue, i.e., there is a surjective local homeomorphism $\mathcal{F}/X \to \mathcal{F}$ with $\mathcal{F}/X$ localic. Using Freyd's result, we not only obtain this theorem directly (with $\mathcal{F}$ a particularly nice étendue), but the fact that $\mathcal{E}$ has a localic cover via an open surjection [**3**, p. 61] also follows. To fix some notation, $\mathrm{sh}(G;B)$ denotes the topos of (discrete) $G$-sheaves, where $B$ is a locale on which the group $G$ acts. $\mathrm{sh}(G;B)$ is a basic example of étendue and is locally $\mathrm{sh}(B)$. Also, $\mathcal{S}^G$ denotes the topos of $G$-sets. There is a canonical map $\mathcal{S}^G \xrightarrow{g} \mathcal{C}(G)$ [**4**].

THEOREM. *Let $\mathcal{E}$ be a Grothendieck topos. There is a hyperconnected geometric morphism $\mathcal{F} \xrightarrow{h} \mathcal{E}$, where $\mathcal{F}$ is an étendue of $G$-sheaves,* $\mathrm{sh}(G;B)$. (*$G$ can be taken to be either the group $G_0$ or $G_1$.*)

PROOF. Let $\mathcal{C}(G)[A] \xrightarrow{f} \mathcal{E}$ be connected and atomic, where $\mathcal{C}(G)[A]$ is the topos of $\mathcal{C}(G)$-sheaves on an internal locale $A$ in $\mathcal{C}(G)$. Let $B$ be the Macneille completion of $A$. Thus $A$ is the cocontinuous reflection of $B$ along the hyperconnected geometric morphism $\mathcal{S}^G \xrightarrow{g} \mathcal{C}(G)$. The following diagram can be seen to be a pullback.

$$\begin{tikzcd} \mathrm{sh}(G;B) \arrow[r, "\overline{g}"] \arrow[d, "\overline{k}"] & \mathcal{C}(G)[A] \arrow[d, "k"] \\ \mathcal{S}^G \arrow[r, "g"] & \mathcal{C}(G) \end{tikzcd}$$

Since pullbacks perserve hyperconnectedness [**3**, p. 54], $\overline{g}$ is hyperconnected. Connected and atomic implies hyperconnected [**2**, Lemma 2.1], thus the composite $\mathrm{sh}(G;B) \xrightarrow{\overline{g}} \mathcal{C}(G)[A] \xrightarrow{f} \mathcal{E}$ is hyperconnected. Take this to be $h$. $\square$

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REMARK. $\mathrm{sh}(G;B) \xrightarrow{\bar{k}} \mathcal{S}^G$ is localic and $\mathcal{S}^G$ is Boolean, thus our étendue satisfies that every object is a quotient of decidable object [**2**, Proposition 4.3].

COROLLARY. *Let $\mathcal{E}$ be a Grothendieck topos. There is an open surjection $\mathrm{sh}(B) \to \mathcal{E}$, where $B$ is a locale on which $G_0$ (or $G_1$) acts.*

PROOF. Compose the hyperconnected map $\mathrm{sh}(G;B) \xrightarrow{h} \mathcal{E}$ with the surjective local homeomorphism $\mathrm{sh}(B) \to \mathrm{sh}(G;B)$. □

Using this open surjection in the Joyal-Tierney representation theorem [**3**, Theorem VIII, 3.2], we have $\mathcal{E}$ is equivalent to a topos of $\Gamma$-sheaves, where $\Gamma$ is a localic groupoid with locale of objects $B$ (on which $G_0(G_1)$ acts).

## REFERENCES


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