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ARCANGELI'S METHOD FOR FREDHOLM EQUATIONS OF THE FIRST KIND

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ABSTRACT. It is well known that a linear operator equation of the first kind, with an operator having nonclosed range, is ill-posed, that is, the solution depends discontinuously on the data. Tikhonov's method for approximating the solution depends on the choice of a positive parameter which effects a trade-off between fidelity and regularity in the approximate solution. If the parameter is chosen according to Morozov's discrepancy principle, then the approximations converge to the true solution as the error level in the data goes to zero. If the operator is selfadjoint and positive and semidefinite, then "simplified" approximations can be formed. We show that Morozov's criterion for the simplified approximations does not result in a convergent method, however, Arcangeli's criterion does lead to convergence. We then prove the uniform convergence of Arcangeli's method for Fredholm integral equations of the first kind with continuous kernel.

1. Introduction. A Fredholm integral equation of the first kind,

$$\int_a^b k(s,t)x(t)\,dt = g(s), \qquad c \leq s \leq d, \tag{1}$$

where $k(\cdot,\cdot)$ is a nondegenerate square integrable kernel, is a well-known example of an ill-posed problem (see e.g. [**7**, **2**]). By this we mean that, even if the solution exists and is unique, the mapping $g \to x$ of the data g to the solution x is discontinuous in the L^2-sense. The discontinuity of the solution operator has dire numerical consequences since in practice the data function g is the result of measurement and hence is only imprecisely known. Small errors in the data can then lead to large instabilities in the computed solution. The method of regularization is designed to ward off these instabilities by replacing (1) with a certain well-posed problem depending on a positive parameter. A central problem is then the choice of this parameter as a function of the error level in the data. Arcangeli's method [**1**] is one strategy for choosing the parameter.

We will find it convenient to deal with the abstract version

$$Kx = g \tag{2}$$

of equation (1), where $K\colon H_1 \to H_2$ is a compact linear operator from a Hilbert space H_1 into a Hilbert space H_2 (the inner product and associated norm in each of the spaces H_1 and H_2 will be denoted $(\cdot,\cdot)$ and $\|\cdot\|$, respectively). We suppose that $g \in R(K)$, the range of K, and that the available data g^δ satisfies $\|g - g^\delta\| \leq \delta$ where δ is an a priori known bound, determined for example by the quality of

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the measuring instruments. Tikhonov proposed to approximate the minimal norm solution x of (2) by the minimizer x_α^δ of the functional

$$F_\alpha(z;g^\delta) = \|Kz - g^\delta\|^2 + \alpha\|z\|^2. \tag{3}$$

Here $\alpha > 0$ is a regularization parameter which effects a trade-off between regularity (small $\|z\|$) and fidelity (small $\|Kz - g^\delta\|$) in the approximate solution. A straightforward calculation shows that the unique minimizer of $F_\alpha(\,\cdot\,;g^\delta)$ is given by

$$x_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*g^\delta$$

where K^* is the adjoint of K. Tikhonov showed that a certain a priori choice $\alpha = \alpha(\delta)$ of α as a function of the error level δ results in $x_{\alpha(\delta)}^\delta \to x$ as $\delta \to 0$.

Morozov amd Arcangeli independently, and at about the same time, provided criteria for the choice of α which depend on the actual data g^δ as well as the error level δ. Morozov's criterion is

$$\|Kx_{\alpha(\delta)}^\delta - g^\delta\| = \delta \tag{4}$$

and Arcangeli's criterion is

$$\|Kx_{\alpha(\delta)}^\delta - g^\delta\| = \delta/\sqrt{\alpha}. \tag{5}$$

In each case it can be shown that a unique such $\alpha(\delta)$ exists and that $x_{\alpha(\delta)}^\delta \to x$ as $\delta \to 0$ (see e.g. [**2**]).

We consider these parameter choice criteria for simplified regularization in this note. For simplified regularization we show that the analogue of Arcangeli's criterion leads to a convergent method, while that of Morozov does not. As an application we prove under suitable conditions the uniform convergence of Arcangeli's method for Fredholm integral equations of the first kind with continuous kernels.

2. Simplified regularization. Consider the equation

$$Aw = g \tag{6}$$

where A is a compact, positive semidefinite linear operator on a Hilbert space H (i.e., $A = A^*$, $(Ax, x) \geq 0$ for all $x \in H$). This equation is also ill-posed and we assume it has a unique solution w. Given approximate data g^δ we may try to approximate w by

$$w_\alpha^\delta = (A + \alpha I)^{-1}g^\delta, \tag{7}$$

the unique minimizer of the functional

$$G_\alpha(z;g^\delta) = (Az, z) - 2(g^\delta, z) + \alpha\|z\|^2.$$

Notice that ordinary Tikhonov regularization applied to (6) results in the more complicated aproximations $(A^2 + \alpha I)^{-1}Ag^\delta$. We therefore refer to (7) as simplified regularization.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ be the eigenvalues of A and let $u_1, u_2, \ldots$ be associated orthonormal eigenvectors. Then

$$\|Aw_\alpha^\delta - g^\delta\|^2 = \sum_{i=1}^\infty \left(\frac{\alpha}{\alpha + \lambda_i}\right)^2 |(g^\delta, u_i)|^2$$

and hence the function

$$\phi(\alpha) = \sqrt{\alpha}\|Aw_\alpha^\delta - g^\delta\|$$

is continuous, increasing, $\phi(0) = 0$, and $\lim_{\alpha\to\infty}\phi(\alpha) = \infty$. Therefore there is a unique positive number $\alpha = \alpha(\delta)$ satisfying

$$\|Aw_{\alpha(\delta)}^\delta - g^\delta\| = \delta/\sqrt{\alpha(\delta)}. \tag{8}$$

THEOREM 1. *If $\alpha = \alpha(\delta)$ is chosen according to* (8), *then $w_{\alpha(\delta)}^\delta \to w$ as $\delta \to 0$.*

PROOF. First we note that $\alpha(\delta) \to 0$ as $\delta \to 0$. This follows by the same argument as in [**2**, Lemma 3.3.7]. To simplify notation we will replace $\alpha(\delta)$ by α in this proof. First note that

$$Aw_\alpha^\delta - g^\delta = (A+\alpha I)(A+\alpha I)^{-1}g^\delta - \alpha w_\alpha^\delta - g^\delta = -\alpha w_\alpha^\delta$$

and hence if α is chosen by (8), then

$$\|w_\alpha^\delta\| = \|Aw_\alpha^\delta - g^\delta\|/\alpha = \delta/\alpha^{3/2}. \tag{9}$$

Let w_α be the idealized approximation using the exact data g:

$$w_\alpha = (A+\alpha I)^{-1}g.$$

Then

$$\|w_\alpha^\delta - w_\alpha\| = \|(A+\alpha I)^{-1}(g^\delta - g)\| \le \delta/\alpha.$$

However, $\|w_\alpha^\delta\| - \|w_\alpha\| \le \|w_\alpha^\delta - w_\alpha\| \le \delta/\alpha$ and hence by (9) $\|w_\alpha^\delta\| \le \|w_\alpha\| + \delta/\alpha = \|w_\alpha\| + \sqrt{\alpha}\delta/\alpha^{3/2} = \|w_\alpha\| + \sqrt{\alpha}\|w_\alpha^\delta\|$. That is, $(1-\sqrt{\alpha})\|w_\alpha^\delta\| \le \|w_\alpha\|$. But, $w_\alpha = (A+\alpha I)^{-1}Aw$ and hence $\|w_\alpha\| \le \|(A+\alpha I)^{-1}A\|\,\|w\| \le \|w\|$. Since $\alpha \to 0$ as $\delta \to 0$, it then follows that

$$\overline{\lim_{\delta\to 0}}\|w_\alpha^\delta\| \le \|w\|. \tag{10}$$

Suppose now that $\{\delta_n\}$ is any sequence of positive numbers convergent to 0. Since $\{w_{\alpha(\delta_n)}^{\delta_n}\}$ is bounded, there is some subsequence, which we again denote $\{w_{\alpha(\delta_n)}^{\delta_n}\}$ which converges weakly to a vector $z \in H$. However,

$$\begin{aligned}\|Aw_{\alpha(\delta_n)}^{\delta_n} - g^{\delta_n}\| &= \delta_n/\sqrt{\alpha(\delta_n)}\\ &= \|w_{\alpha(\delta_n)}^{\delta_n}\|\alpha(\delta_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore by the weak continuity of A,

$$0 = \lim_{n\to\infty}(Aw_{\alpha(\delta_n)}^{\delta_n} - g^{\delta_n}) = Az - g,$$

i.e., $Az = g$ and hence $z = w$. Also

$$\|w\|^2 = \lim_{n\to\infty}(w, w_{\alpha(\delta_n)}^{\delta_n}) \le \|w\| \underline{\lim}_{n\to\infty} \|w_{\alpha(\delta_n)}^{\delta_n}\|.$$

From (10) we then have $\lim_{n\to\infty}\|w_{\alpha(\delta_n)}^{\delta_n}\| = \|w\|$. But in a Hilbert space weak convergence along with convergence of norms implies strong convergence, therefore $w_{\alpha(\delta_n)}^{\delta_n} \to w$ as $n \to \infty$. Hence for each sequence $\delta_n \to 0$, there is a subsequence of $\{w_{\alpha(\delta_n)}^{\delta_n}\}$ converging to w. That is, $w_{\alpha(\delta)}^\delta \to w$ as $\delta \to 0$. □

The argument above can be modified in a straightforward manner to prove convergence for certain analogues of the discrepanacy principle studied by Schock [**6**].

It is also easy to show that if $\|g-g^\delta\| \le \delta < \|g^\delta\|$, then there is a unique positive $\alpha = \alpha(\delta)$ such that

$$\|Aw^\delta_{\alpha(\delta)} - g^\delta\| = \delta. \tag{11}$$

This choice of α is the analogue of Morozov's method for simplified regularization. However the choice (11) does not always lead to a convergent method for solving (6). To see this suppose that $1 = \lambda_1 > \lambda_2 \ge \cdots$ are the positive eigenvalues of A and let $u_1, u_2, \ldots$ be corresponding orthonormal eigenvectors. Let $g - u_1$, so that $w = u_1$. Also let $\delta_n = \lambda_n$ and $g^{\delta_n} = u_1 + \lambda_n u_n$. Then

$$\begin{aligned} w^{\delta_n}_\alpha &= (A+\alpha I)^{-1} g^{\delta_n} \\ &= \sum_{i=1}^{\infty} \frac{(g^{\delta_n}, u_i)}{\lambda_i + \alpha} u_i = \frac{1}{1+\alpha} u_1 + \frac{1}{1+\alpha/\delta_n} u_n. \end{aligned} \tag{12}$$

However,

$$Aw^{\delta_n}_\alpha - g^{\delta_n} = (A+\alpha I) w^{\delta_n}_\alpha - g^{\delta_n} - \alpha w^{\delta_n}_\alpha = -\alpha w^{\delta_n}_\alpha,$$

and hence if $\alpha(\delta_n)$ is chosen by the criterion (11), then $\delta_n = \alpha(\delta_n)\|w^{\delta_n}_{\alpha(\delta_n)}\|$. In particular, if $w^{\delta_n}_{\alpha(\delta_n)} \to w = u_1$, then $\alpha(\delta_n) = O(\delta_n)$. But, by (12), $w^{\delta_n}_{\alpha(\delta_n)} \to u_1$ if and only if $\alpha(\delta_n)/\delta_n \to \infty$, contradicting the fact that $\alpha(\delta_n) = O(\delta_n)$.

3. Uniform convergence. We now use the theorem of the previous section to establish the uniform convergence of Arcangeli's method for equation (1) under the assumption that the unique solution x lies in $R(K^*)$ and the kernel $k(\cdot,\cdot)$ is continuous. The assumption $x \in R(K^*)$ has been used by Landweber [**5**] to prove the uniform convergence of an iterative method for first kind equations with continuous kernels in the absence of data errors (see also [**3**]). The same assumption is used in [**4**] to provide an asymptotic order of convergence for Arcangeli's method in Hilbert space.

We take as our base Hilbert spaces $H_1 = L^2[a,b]$ and $H_2 = L^2[c,d]$. The L^2-norm and inner product will be denoted $\|\cdot\|$ and $(\cdot,\cdot)$, respectively, and the uniform norm will be denoted $\|\cdot\|_\infty$. Note that each of the approximations x^δ_α is in $R(K^*)$ and hence is continuous since the operator K^* is generated by a continuous kernel (we assume that $g^\delta \in L^2[c,d]$, but not that it is continuous). Also,

$$Kx^\delta_\alpha = KK^*(KK^* + \alpha I)^{-1} g^\delta = Aw^\delta_\alpha,$$

where $A = KK^*$ and $w^\delta_\alpha = (A+\alpha I)^{-1} g^\delta$. If α is chosen by Arcangeli's method (5), then $\|Aw^\delta_\alpha - g^\delta\| = \|Kx^\delta_\alpha - g^\delta\| = \delta/\sqrt{\alpha}$ and hence criterion (8) is satisfied. Therefore, by Theorem 1, $\|w^\delta_\alpha - w\| \to 0$ as $\delta \to 0$ where $x = K^* w$.

THEOREM 2. *Suppose $k(\cdot,\cdot)$ is continuous and $x = K^* w$ for a unique $w \in L^2[c,d]$. If $\alpha = \alpha(\delta)$ is chosen according to* (5), *then $\|x^\delta_{\alpha(\delta)} - x\|_\infty \to 0$ as $\delta \to 0$.*

PROOF. For any $s \in [a,b]$ we have

$$x(s) = (K^* w)(s) = \int_c^d k(t,s) w(t)\, dt = (k_s, w)$$

and

$$x^\delta_\alpha(s) = (K^* w^\delta_\alpha)(s) = (k_s, w^\delta_\alpha)$$

where $k_s(t) = k(t,s)$. Since the kernel $k(\cdot,\cdot)$ is continuous, we have, for some constant M,

$$|x(s) - x_\alpha^\delta(s)| = |(k_s, w - w_\alpha^\delta)| \leq M\|w - w_\alpha^\delta\|,$$

and hence

$$\|x - x_\alpha^\delta\|_\infty \leq M\|w - w_\alpha^\delta\| \to 0 \quad \text{as } \delta \to 0. \quad \square$$

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