PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

THE OSELEDEC AND SACKER-SELL SPECTRA FOR ALMOST PERIODIC LINEAR SYSTEMS: AN EXAMPLE

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ABSTRACT. We give an example illustrating the relation between the Oseledec spectrum (roughly speaking, the set of Lyapunov exponents) and the Sacker-Sell (or continuous) spectrum for Bohr almost periodic linear systems.

1. Introduction. The purpose of this note is to illustrate by means of an example the relation between the Oseledec (or measurable) spectrum [**10**] and the Sacker-Sell (or continuous) spectrum [**12, 13**] for Bohr almost periodic linear systems

$$x' = A(t)x \qquad (x \in \mathbf{R}^k). \tag{1}$$

Our example complements a result of [**7**], according to which an endpoint of an interval in the continuous spectrum is necessarily in the measurable spectrum. In fact, equation (1) which we construct has a point in its measurable spectrum which is an interior point of an interval in the continuous spectrum. In itself this property is of no great significance; however, our equation enjoys an additional property of "irreducibility": if $x_1(t)$, $x_2(t)$ are two nonzero solutions of (1), then there is a sequence $\{t_n\}$ such that the angle θ_n between $x_1(t_n)$ and $x_2(t_n)$ tends to zero or π as $n \to \infty$.

Let us explain some of the terminology just used. One usually defines Bohr almost periodicity using translation numbers, but it is convenient here to adopt another starting point. Thus let $\mathcal{C}$ be the space of bounded, uniformly continuous maps B from $\mathbf{R}$ to $M_k(\mathbf{R})$ = set of $k \times k$ real matrices. Give $\mathcal{C}$ the topology of uniform convergence on all of $\mathbf{R}$. Define the translation $\tau_t\colon \mathcal{C} \to \mathcal{C}\colon (\tau_t B)(s) = B(t+s)$ $(B \in \mathcal{C}; t, s \in \mathbf{R})$. If $B \in \mathcal{C}$, define the *hull* $Y = Y_B$ of B to be $\operatorname{cls}\{\tau_t(B) | t \in \mathbf{R}\}$. We say that B is *Bohr almost periodic* (a.p.) if Y is compact. It turns out that, in this case, Y may be given the structure of a compact, abelian topological group with identity B and multiplication $*$ satisfying $\tau_t(B) * \tau_s(B) = \tau_{t+s}(B)$ for all $s, t \in \mathbf{R}$. Thus the map $t \to \tau_t(B)$ defines a dense imbedding of the additive group $(\mathbf{R}, +)$ in $(Y, *)$.

Suppose now that $A \in \mathcal{C}$ is a.p., with hull Y. Let μ be normalized Haar measure on Y. Consider the equations

$$x' = \tilde{A}(t)x \qquad (\tilde{A} \in Y,\ x \in \mathbf{R}^k). \tag{$\tilde{1}$}$$

The Oseledec theorem [**10**] tells us, among other things, that there is a set $\Sigma_m = \{\beta_1, \ldots, \beta_k\}$ of real numbers with $\beta_1 \le \beta_2 \le \cdots \le \beta_k$ such that, for μ-a.a. $\tilde{A} \in Y$, the equation $(\tilde{1})$ has linearly independent solutions $x_1(t), \ldots, x_k(t)$ satisfying

Received by the editors December 5, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C35, 54H20.
Key words and phrases. Almost periodic, minimal set, Lyapunov number.

$\lim_{t\to\infty}, \lim_{t\to-\infty} t^{-1}\ln\|x_k(t)\| = \beta_k$. Also, $\beta_1+\cdots+\beta_k = \lim_{t\to\infty} t^{-1}\int_0^t \operatorname{tr} A(s)\,ds$. In addition, if $\tilde{\Phi}(t)$ is the fundamental matrix solution of ($\tilde{1}$) satisfying $\tilde{\Phi}(0) = I$, then

$$\lim_{t\to\infty} t^{-1}\ln\|\tilde{\Phi}(t)\| = \beta_k, \qquad \lim_{t\to-\infty} t^{-1}\ln\|\tilde{\Phi}(t)\| = \beta_1.$$

The β_i's are called *Lyapunov numbers*, and are independent of $\tilde{A}$ for μ-a.a. $\tilde{A}\in Y$. The set Σ_m is the Oseledec or measurable spectrum of A.

Next we consider the Sacker-Sell or continuous spectrum Σ_c of A. Recall first that an equation

$$x' = B(t)x \qquad (x\in \mathbf{R}^k) \tag{2}$$

has *exponential dichotomy* if there exist constants $K>0$, $\alpha>0$, and a projection $Q\colon \mathbf{R}^k\to\mathbf{R}^k$ such that, if $\Psi(t)$ is the fundamental matrix solution of (2) satisfying $\Psi(0)=I$, then

$$\begin{aligned}\|\Psi(t)Q\Psi^{-1}(s)\| &\le Ke^{-\alpha(t-s)} \qquad (t\ge s),\\ \|\Psi(t)(I-Q)\Psi^{-1}(s)\| &\le Ke^{-\alpha(s-t)} \qquad (t\le s)\end{aligned}$$

(see, e.g., Coppel [**1**]). Define $\Sigma_c = \{\lambda\in\mathbf{R}\,|\,x' = (A(t)-\lambda I)x$ does *not* have exponential dichotomy$\}$ (see [**12, 13**]). It is known that Σ_c is a finite union of (at most k) compact intervals, and that, for any $\tilde{A}\in Y$ and any nonzero solution $x(t)$ of ($\tilde{1}$),

$$\overline{\lim_{t\to\infty}},\ \overline{\lim_{t\to-\infty}}\ t^{-1}\ln\|x(t)\| \quad\text{and}\quad \underline{\lim_{t\to\infty}},\ \underline{\lim_{t\to-\infty}}\ t^{-1}\ln\|x(t)\|$$

all belong to Σ_c. That is, all upper and lower Lyapunov numbers are in Σ_c.

Clearly it is of interest to compare Σ_m and Σ_c. As we have already noted, an endpoint of an interval in Σ_c is in Σ_m [**7**]. For 2×2 systems (1), Millionshchikov [**8**] showed that Σ_m is exactly the set of endpoints of intervals in Σ_c. This is no longer true for three-dimensional systems; in fact we may take

$$A(t) = \begin{pmatrix} A_1(t) & 0\\ 0 & 0\end{pmatrix},$$

where $A_1(t)$ is 2×2 with $\Sigma_c(A_1) = [-\gamma,\gamma]$, and $\gamma>0$ [**9**]. Then clearly $\{\beta_1,\beta_2,\beta_3\} = \{-\gamma,0,\gamma\}$. However, if $A(t)$ is not reducible to a system in block-form, then the situation is less clear, and one might conjecture that, if (1) is (in some sense) irreducible, then Σ_m is the set of endpoints of intervals in Σ_c.

The example we will construct has the property that $\Sigma_c = [b_1, b_2]$ but $\Sigma_m = \{b_1, a, b_2\}$ with $b_1 < a < b_2$. To explain just how irreducible the example is, we introduce the corresponding projective flow. Thus let $\mathbf{P}^2(\mathbf{R})$ be the projective space of all lines l through the origin in R^3. Let $\mathbf{P} = Y\times\mathbf{P}^2(\mathbf{R})$. Define

$$\hat{\tau}_t(\tilde{A}, l) = (\tau_t(\tilde{A}), \tilde{\Phi}(t)l) \qquad (t\in\mathbf{R}),$$

where $\tilde{\Phi}(t)l$ is the image of l under $\tilde{\Phi}(t)$ ($l\in\mathbf{P}^2(\mathbf{R}), \tilde{A}\in Y$). Then $\{\hat{\tau}_t | t\in\mathbf{R}\}$ defines a flow [**3**] on $\mathbf{P}$. It turns out that $\mathbf{P}$ is a *proximal extension* of Y: if $l_1,\ l_2\in\mathbf{P}^2(\mathbf{R})$ and $\tilde{A}\in Y$, then there is a sequence $\{t_n\}$ such that distance $(\tilde{\Phi}(t_n)l_1, \tilde{\Phi}(t_n)l_2)\to 0$ as $n\to\infty$. This is the property referred to in the first paragraph. More is true: $\mathbf{P}$ contains a unique minimal set [**3**] M, which is an *almost-automorphic extension of* Y [**5**, **14**]. This means that, for y in a residual subset $Y_0 \subset Y$, the fiber $M_y = M \cap (\{y\} \times \mathbf{P}^2(\mathbf{R}))$ reduces to a single point. Thus our example differs from those constructed in [**4**], where $M = \mathbf{P}$, and Σ_m is a single point.

The moral, then, is that A is trying simultaneously to be nilpotent (M is almost automorphic) and to have three distinct real eigenvalues (Σ_c consists of three distinct points).

2. The example. The construction to follow is basically an elaboration of one of Millionshchikov [**9**]. The desired matrix function $A(t)$ will be a uniform limit of T_n-periodic, continuous matrix functions $A_n(t)$, where $T_{n+1} = j_n T_n$ for whole numbers j_n $(n = 1, 2, \ldots)$. Thus $A(t)$ will be limit-periodic.

First of all, we fix some notation. If $0 \neq x \in \mathbf{R}^3$, let $[x]$ be the unit vector $x/\langle x, x\rangle^{1/2} \in S^2$. Let $\varepsilon_0 = 0$, and let $\{\varepsilon_n | n \geq 1\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n = 1$. We will consider systems

$$(1)_n \qquad x' = A_n(t)x \qquad (x \in \mathbf{R}^3);$$

let $\Phi_n(t)$ be the fundamental matrix solution of $(1)_n$ which satisfies $\Phi_n(0) = I$.

Let $A_1(t)$ be a continuous matrix function of period $T_1 \geq 4$ with the following properties:

$$(3)_1 \qquad A_1(t) = 0 \qquad (t \in [0, 1] \cup [T_1 - 2, T_1]);$$

$$(4)_1 \qquad \operatorname{tr} A_1(t) \equiv 0.$$

Suppose furthermore that there are linearly independent unit vectors u_1, v_1, w_1 such that

$$(5)_1 \qquad \begin{aligned} \Phi_1(T_1)u_1 &= (\exp -5T_1)u_1, \\ \Phi_1(T_1)v_1 &= v_1, \\ \Phi_1(T_1)w_1 &= (\exp 5T_1)w_1, \end{aligned}$$

and

$$(6)_1 \qquad \theta(u_1, v_1) < (\varepsilon_1)^2, \qquad \theta(u_1, w_1) < (\varepsilon_1)^2.$$

Here $\theta(a, b)$ denotes the angle between the unit vectors a and b; we always suppose that $0 \leq \theta(a, b) < \pi$ radians. Finally, assume that

$$(7)_1 \qquad -5.1 \leq \frac{1}{t - s} \ln \|\Phi_1(t)\Phi_1^{-1}(s)\| \leq 5.1 \qquad (t \neq s).$$

Clearly a function $A_1(t)$ can be found for which $(3)_1$–$(7)_1$ are valid.

Now suppose that A_n is a continuous matrix function of period $T_n \geq 4$ such that

$$(3)_n \qquad A_n(t) = 0 \qquad \left(t \in [0, 1] \cup \left[T_n - 2 + \sum_{i=1}^{n-1} \varepsilon_i, T_n\right]\right),$$

$$(4)_n \qquad \operatorname{tr} A_n(t) \equiv 0.$$

Assume in addition that u_n, v_n, w_n are linearly independent unit vectors such that

$$(5)_n \qquad \begin{aligned} &\Phi_n(T_n)u_n = (\exp -\gamma_n T_n)u_n, \\ &\Phi_n(T_n)v_n = v_n, \\ &\Phi_n(T_n)w_n = (\exp \gamma_n T_n)w_n, \qquad \gamma_n \geq 5 - \sum_{i=1}^{n-1} \varepsilon_i; \end{aligned}$$

$$(6)_n \qquad \theta(u_n, v_n) < \varepsilon_n^2, \qquad \theta(u_n, w_n) < \varepsilon_n^2;$$

$$(7)_n \qquad -5.1 \leq \frac{1}{t-s} \ln \|\Phi_n(t)\Phi_n^{-1}(s)\| \leq 5.1 \qquad (t \neq s).$$

Our goal is to find a continuous matrix function $A_{n+1}(t)$ and an integer $j_n \geq 1$ such that A_{n+1} has period $T_{n+1} = j_n T_n$, and such that

$$(8)_{n+1} \qquad \|A_{n+1}(t) - A_n(t)\| < 4\varepsilon_n \qquad (t \in \mathbf{R}).$$

To begin, let P_n be the plane in $\mathbf{R}^3$ spanned by u_n and w_n. Using a technique of Millionshchikov (to be described shortly), we can choose an integer $J_1 \geq 1$ such that, if $j \geq J_1$, then a rotation R of $\mathbf{R}^3$ can be found which preserves P_n, fixes vectors normal to P_n, satisfies $\|R - I\| < 2\varepsilon_n^2$, and has the property that the matrix $H = R \circ \Phi_n(jT_n)$ fulfills the following conditions:

(9) There are unit vectors $u_{n+1}, w_{n+1} \in P_n$ which are "between" (in the obvious sense) u_n and w_n such that $Hu_{n+1} = (\exp -\gamma j T_n)u_{n+1}$ and $Hw_{n+1} = \exp(\gamma j T_n)w_{n+1}$ with $\gamma \geq 5 - \sum_{i=1}^{n} \varepsilon_i$;

$$(10) \qquad \theta(u_n, u_{n+1}) < \theta(u_n, w_{n+1}) < \tfrac{1}{3}\varepsilon_{n+1}^2 \min(1, \theta(u_n, w_n)/\theta(u_n, v_n));$$

$$(11) \qquad \theta(u_n, w_{n+1}) > \tfrac{1}{6}\varepsilon_{n+1}^2 \min(1, \theta(u_n, w_n)/\theta(u_n, v_n)).$$

We briefly indicate how J_1, R, and the vectors u_{n+1}, w_{n+1} may be found. Let $\Gamma(t)$ be the restriction of $\Phi_n(t)$ to P_n. Then $\Gamma(T_n)$ is a linear mapping of P_n to itself which has eigenvectors u_n, w_n with eigenvalues δ^{-1}, δ respectively, where $\delta = \exp \gamma_n T_n$ and $\gamma_n \geq 5 - \sum_{i=1}^{n-1} \varepsilon_i$. Consider unit vectors $x \in P_n$ which are between u_n and w_n. If $x \neq u_n$, then $\lim_{j\to\infty}[\Gamma(jT_n) \cdot x] = W_n$; i.e., x is rotated towards w_n as j increases. Moreover $\lim_{j\to\infty}(1/jT_n) \ln \|\Gamma(jT_n)x\| = \gamma_n$.

Let $\tilde{\theta}_j(x)$ be the angle between u_n and $[\Gamma(jT_n)x]$. Then $\tilde{\theta}_j(u_n) = 0$, and $\lim_{j\to\infty} \tilde{\theta}_j(x) = \theta(u_n, w_n)$ if $x \neq u_n$. For any $0 < r < \theta(u_n, w_n)$, let R be the rotation of the plane P_n which displaces w_n towards u_n by r radians. Then for large j, there will be exactly two vectors between u_n and w_n which are eigenvectors of $R \circ \Gamma(jT_n)$. If we choose

$$\begin{aligned} &\theta(u_n, w_n) - \tfrac{1}{3}\varepsilon_{n+1}^2 \min(1, \theta(u_n, w_n)/\theta(u_n, v_n)) \\ &\qquad < r < \theta(u_n, w_n) - \tfrac{1}{6}\varepsilon_{n+1}^2 \min(1, \theta(u_n, w_n)/\theta(u_n, v_n)), \end{aligned}$$

then, for large j, (9)–(11) will hold for the eigenvectors u_{n+1}, w_{n+1} of $R \circ \Gamma(jT_n)$ (see [**9**], also [**6**, §5]).

Returning to the construction of A_{n+1}, fix $j \geq J_1$. Let b be a continuous map from $\mathbf{R}$ to the set of antisymmetric, real, 3×3 matrices such that b vanishes outside of $[0, \varepsilon_n]$, $\sup_t \|b(t)\| \leq 4\varepsilon_n$, and such that the 3×3 matrix solution of

$$\eta' = b(t)\eta, \qquad \eta(0) = I,$$

satisfies $\eta(\varepsilon_n) = R$. Consider the function

$$B(t) = \begin{cases} A_n(t), & 0 \leq t \leq jT_n - 2 + \sum_{i=1}^{n-1} \varepsilon_i, \\ b\left(t - \left[jT_n - 2 + \sum_{i=1}^{n-1} \varepsilon_i\right]\right), & jT_n - 2 + \sum_{j=1}^{n-1} \varepsilon_i \leq t \leq jT_n; \end{cases}$$

we extend $B(t)$ to all of $\mathbf{R}$ by jT_n-periodicity. Let $\Psi(t)$ be the fundamental matrix solution of $x' = B(t)x$ which satisfies $\Psi(0) = I$. Then $\Psi(jT_n) = H = R \circ \Phi_n(jT_n)$.

We claim that, if j is sufficiently large, we can set $A_{n+1}(t) = B(t)$ and fulfill all conditions $(3)_{n+1}$–$(8)_{n+1}$. In fact, $(3)_{n+1}$, $(4)_{n+1}$, $(7)_{n+1}$, and $(8)_{n+1}$ are true for any $j \geq J_1$, if $A_{n+1} = B$. Moreover, for $j \geq J_1$, the vectors u_{n+1} and w_{n+1} satisfy the corresponding parts of $(5)_{n+1}$ and $(6)_{n+1}$. Using $(4)_n$, (9), and Liouville's formula, we see that there is a unit vector $v_{n+1} \in \mathbf{R}^3$ such that $Hv_{n+1} = v_{n+1}$; i.e., $(5)_{n+1}$ is completely satisfied. So we need only show that j and v_{n+1} can be chosen in such a way that the first part of $(6)_{n+1}$ holds.

To do so, once again fix $j \geq J_1$. Consider the spherical triangle Δ_n with vertices u_n, v_n, w_n; thus $\Delta_n \subset S^2$. The vector w_{n+1} lies on the side $u_n w_n$ (i.e., arc of the great circle containing u_n and w_n) of Δ_n. Let p be a general point on the side $u_n v_n$ of Δ_n. Let φ_p be the measure, in radians, of the spherical angle with vertex w_{n+1} and sides $u_n w_{n+1}$, $u_n p$.

Let $\varphi_p(t)$ be the measure of the spherical angle with vertex $[\Phi_n(t)w_{n+1}]$ and sides determined by $[\Phi_n(t)u_n]$, $[\Phi_n(t)p]$. Note that, if $p \neq u_n$, then $\lim_{j\to\infty} \varphi_p(jT_n) = \varphi_\infty =$ measure of the angle with sides $u_n w_n$ and $w_n v_n$. Hence by (10) and spherical trigonometry, one has that, if $\theta(u_n, p) > \frac{1}{3}\varepsilon_{n+1}^2$, then for large j, $\varphi_p(jT_n) > \varphi_p(0)$. On the other hand, if $\theta(u_n, p)/\theta(u_n, v_n) < \theta(u_n, w_{n+1})/\theta(u_n, w_n)$, then for large j, $\varphi_p(jT_n) < \varphi_p(0)$. (Note that, by (11), $\theta(u_n, w_{n+1})/\theta(u_n, w_n) \geq c > 0$, where c is independent of j.) Thus there exists $J_2 \geq J_1$ such that, if $j \geq J_2$, then there is a point p_0 with $\theta(u_n, p_0) < \frac{1}{3}\varepsilon_{n+1}^2$ and $\varphi_{p_0}(jT_n) = \varphi_{p_0}(0)$. In particular, the plane W_n spanned by w_{n+1} and p_0 is invariant under $H = R \circ \Phi_n(jT_n)$.

Fix $j \geq J_2$. We know that there is a unit vector v_{n+1} such that $Hv_{n+1} = v_{n+1}$. We also know that $\theta(u_n, p_0) < \frac{1}{3}\varepsilon_{n+1}^2$. Hence we can show that the first part of $(6)_{n+1}$ holds by proving that v_{n+1} can be chosen to lie on the arc $\sigma = w_{n+1}p_0$. To do so, note that, if $x \in \sigma$ is close to w_{n+1}, then $[Hx]$ is even closer to w_{n+1}. This is because w_{n+1} is an eigenvector corresponding to the largest eigenvalue of H. On the other hand, the arc $\{[Hx] | x \in \sigma\}$ is longer than σ, hence $[Hp_0]$ is further away from w_{n+1} than is p_0 itself. Thus there is a point $x_0 \in \sigma$ such that $[Hx_0] = x_0$, and we can take $v_{n+1} = x_0$.

We have shown that, if $j \geq J_2$, and if $A_{n+1}(t) = B(t)$ $(t \in \mathbf{R})$, then $(3)_{n+1}$–$(8)_{n+1}$ are satisfied by the system $(1)_{n+1}$.

It will be convenient to impose a further condition on j. Namely, let $U_n \subset S^2$ be an open set containing the arc $u_n v_n$ such that diameter $U_n < \varepsilon_n$ and $w_n \notin U_n$.

We can choose $J_3 \geq J_2$ such that, for $j \geq J_3$:

$(12)_{n+1}$ if $\pm x \notin U_n$, then either $\theta(Hx, w_{n+1}) < \varepsilon_{n+1}^2$ or $\theta(Hx, -w_{n+1}) < \varepsilon_{n+1}^2$.

Now fix $j_n \geq J_3$, and let $T_{n+1} = j_n T_n$, $A_{n+1} = B$. By induction, we obtain a sequence of matrices $A_1, A_2, \ldots$, all periodic with periods $T_1, T_2 = j_1 T_1, T_3 = j_2 T_2$, etc. Conditions $(3)_n$–$(8)_n$ $(n \geq 1)$ and $(12)_n$ $(n \geq 2)$ hold for the corresponding systems $(1)_n$. Let $A(t) = \lim_{n\to\infty} A_n(t)$, so that $A(t)$ is limit-periodic (see $(8)_n$). Let Y be the hull of A, and consider the equations

$$(\tilde{1}) \qquad x' = \tilde{A}(t)x \qquad (\tilde{A} \in Y).$$

Let Σ_c be the continuous spectrum of equation (1). We claim that Σ_c is a single interval. For, if not, there would exist a projection $Q\colon \mathbf{R}^3 \to \mathbf{R}^3$ and real numbers $\alpha_1 \leq \alpha_2 < \alpha_3 \leq \alpha_4$ such that: (i) $Q \neq 0$, $Q \neq I$; (ii) if $0 \neq x = Qx$, then $\overline{\lim}_{t\to\pm\infty}, \underline{\lim}_{t\to\pm\infty} t^{-1} \ln \|\Phi(t)x\| \in [\alpha_1, \beta_1]$; (iii) if $0 \neq x = (I - Q)x$, then $\overline{\lim}_{t\to\pm\infty}, \underline{\lim}_{t\to\pm\infty} t^{-1} \ln \|\Phi(t)x\| \in [\alpha_2, \beta_2]$ (see [**12, 13**]). Now we use a perturbation theorem of Coppel [**1**] to conclude that, for large n, there are projections $Q_n \to Q$ and constants α_i^n $(1 \leq i \leq 4)$ such that the above statements hold with $\Phi_n(t)$ in place of $\Phi(t)$. It is easily seen, however, that the range of Q_n must be a sum of eigenspaces of $\Phi_n(T_n)$. Thus $Q_n \to Q$ is inconsistent with $(6)_n$ for large n, and we conclude that Σ_c is indeed a single interval.

Let $\Sigma_c = [b_1, b_2]$ with $b_1 \leq b_2$. Let $\Sigma_m = \{\beta_1, \beta_2, \beta_3\}$ $(\beta_1 \leq \beta_2 \leq \beta_3)$ be the measurable spectrum of equations $(\tilde{1})$. Then $\beta_1 = b_1$, $\beta_3 = b_2$ [**7**]. It follows from $(7)_n$ of our construction that $\|\Phi(t)\Phi^{-1}(s)\| \leq \exp 5.1(t - s)$ for all $t \geq s$. Hence $\ln \|\tilde{\Phi}(t)\tilde{\Phi}^{-1}(s)\| \leq 5.1(t - s)$ for $t - s \geq 0$, for all $\tilde{A} \in Y$; we use the fact that $\{\tau_t(A) | t \in \mathbf{R}\}$ is dense in Y. Hence $b_2 \leq 5.1$. Similarly $b_1 \geq -5.1$. However, we also know that $b_2 \geq \overline{\lim}_{t\to\infty} t^{-1} \ln \|\tilde{\Phi}(t)\|$ (e.g., [**2, 7**]). Hence by $(5)_n$, $b_2 \geq 4$. Similarly $b_1 \leq -4$. Since $\operatorname{tr} A(t) \equiv 0$, we have $\beta_1 + \beta_2 + \beta_3 = 0$. It is now clear that $\beta_2 \neq \beta_1$, $\beta_2 \neq \beta_3$. Thus Σ_m consists of three distinct numbers.

Next we consider irreducibility properties of equation (1). We show first that $\mathbf{P} = Y \times \mathbf{P}^2(\mathbf{R})$ contains a unique minimal set M which is an almost automorphic extension of the base Y. To do so, fix attention on $A \in Y$. Let Δ_n be the spherical triangle with vertices u, v, w $(n \geq 1)$, and let $\{x_0\} = \bigcap_{n=1}^{\infty} \Delta_n$. Let $\{s_k\}$ be a sequence such that $\tau_{s_k}(A) \to A$. Using the duality theory of compact, abelian topological groups [**11**], one can show that, for fixed n, $s_k \bmod T_n \to 0$ as $k \to \infty$. Fix n, and choose $K = K(n)$ such that, if $k \geq K$, then $|s_k \bmod T_n| \leq \frac{1}{2}$. It follows from our construction that, if $m \geq n$ and $j \geq 0$, then $\{[\Phi_m(jT_n)x] | x \in \Delta_m\} \subset \Delta_n$ (see $(5)_n$, (9), and note that $\Delta_{n+1} \subset \Delta_n$ for $n \geq 1$). Since $A(t) = A_m(t)$ for $0 \leq t \leq T_m$, and since $A_m(t) = 0$ for $T_m - 1 \leq t \leq T_{m+1}$ $(m \geq 1)$, we see that $[\Phi(s_k)x_0] \in \Delta_n$ for $k \geq K$. Projecting from S^2 to $\mathbf{P}^2(\mathbf{R})$, and recalling from §1 the definition of the flow $\hat{\tau}$ on $\mathbf{P}$, we conclude that $\hat{\tau}_{s_k}(A, l_0) \to (A, l_0)$ as $k \to \infty$. Here $l_0 \in \mathbf{P}^2(\mathbf{R})$ is the line containing x_0.

Let $M = \operatorname{cls}\{\tau_t(A, l_0) | t \in \mathbf{R}\} \subset \mathbf{P}$. Then $M \cap (\{A\} \times \mathbf{P}^2(\mathbf{R}))$ equals $\{(A, l_0)\}$, i.e. is a singleton. It follows without difficulty that M is a minimal, almost automorphic extension of Y (see, e.g., [**14**]).

To prove uniqueness of M, suppose that $M_1 \neq M$ is another minimal subset of $\mathbf{P}$. Then $M_1 \cap M = \varnothing$. Let $(A, l_1) \in M_1$, and let $0 \neq x$ be an element of the line l_1. Then $[\Phi(t)x]$ is bounded away from $[\Phi(t)x_0]$, uniformly in $t \in \mathbf{R}$. This contradicts $(12)_{n+1}$ and the fact that $A(t) = A_n(t)$ for $0 \leq t \leq T_n$, when n is taken large.

Finally, we show that $\mathbf{P}$ is a proximal extension of Y. By minimality of Y, it is sufficient to show that, if $l_1, l_2 \in \mathbf{P}^2(\mathbf{R})$, then there is a sequence $t_k \to \infty$ such that distance $(\Phi(t_k)\cdot l_1, \Phi(t_k)\cdot l_2) \to 0$ as $k \to \infty$. However, this follows from conditions $(12)_{n+1}$ and the relation $A(t) = A_n(t)$ $(0 \le t \le T_n,\ n \ge 1)$. Thus we have shown that (1) has all the properties set out for it in the Introduction.

It is worth noting that, by our construction, the planes $\{P_n\}$ satisfy $P_1 = P_2 = P_3 = \cdots$. If we choose A_1 in such a way that P_1 is invariant under $\Phi_1(t)$ (as we certainly can do), then M is in fact a subset of $Y \times P_1$.

REFERENCES

1. A. Coppel, *Dichotomies and stabilitity theory*, Lecture Notes in Math., vol. 629, Springer-Verlag, Berlin and New York, 1978.
2. J. Daletskii and M. Krein, *Stability of solutions of differential equations in Banach space*, Transl. Math. Monos., vol. 43, Amer. Math. Soc., Providence, R.I., 1974.
3. R. Ellis, *Lectures on topological dynamics*, Benjamin, Reading, Mass., 1969.
4. R. Ellis and R. Johnson, *Topological dynamics and linear differential systems*, J. Differential Equations **44** (1982), 21–39.
5. H. Furstenberg, H. Keynes and L. Shapiro, *Prime flows in topological dynamics*, Israel J. Math. **14** (1973), 26–38.
6. R. Johnson, *The recurrent Hills equation*, J. Differential Equations **46** (1982), 165–194.
7. R. Johnson, K. Palmer and G. Sell, *Ergodic theory of linear dynamical systems*, SIAM J. Appl. Math. (to appear).
8. V. Millionshchikov, *Metric theory of linear systems of differential equations*, Math. USSR-Sb. **6** (1968), 149–158.
9. ——, *Proof of the existence of non-irreducible systems of linear differential equations with almost periodic coefficients*, J. Differential Equations **4** (1968), 203–205.
10. V. Oseledec, *A multiplicative ergodic theorem*, Trans. Moscow Math. Soc. **19** (1968), 197–231.
11. L. Pontryagin, *Topologische Gruppen*, Deutsche Ubersetzung, Teubner Verlag, Leipzig, 1957.
12. R. Sacker and G. Sell, *Dichotomies and invariant splittings for linear differential systems.* I, J. Differential Equations **15** (1974), 429–458.
13. ——, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.
14. W. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751.

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