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JENSEN'S INEQUALITY FOR POSITIVE CONTRACTIONS ON OPERATOR ALGEBRAS

DÉNES PETZ

ABSTRACT. Let τ be a normal semifinite trace on a von Neumann algebra, and let f be a continuous convex function on the interval $[0,\infty)$ with $f(0)=0$. For a positive element a of the algebra and a positive contraction α on the algebra, the following inequality is obtained:

$$\tau(f(\alpha(a))) \leq \tau(\alpha(f(a))).$$

Versions of Jensen's inequality play in important role in several parts of mathematics and applications. One of the standard tools in quantum statistical mechanics is the Peierls-Bogoliubov inequality

$$\operatorname{tr} f(A) \geq \sum_i f(\langle A\eta_i, \eta_i\rangle),$$

where f is a convex function on the real line, A is a selfadjoint operator, and $\{\eta_i\}$ is an orthonormal set (see [**11**], for example). Slightly more general, if $\{P_i\}$ is a pairwise orthogonal family of projections and we set $\alpha(A) = \sum_i P_i A P_i$ for a selfadjoint operator A, then

$$\operatorname{tr} \alpha(f(A)) \geq \operatorname{tr} f(\alpha(A))$$

(see [**2**, **6**, and **9**]). Going further in this direction Brown and Kosaki [**3**] have recently proved that

$$\tau(f(v^*av)) \leq \tau(v^* f(a) v),$$

where τ is a faithful normal semifinite trace on a von Neumann algebra, v is a contraction, and a is a positive element of the algebra. (In fact, a may be unbounded and affiliated with the algebra.) In the present paper we obtain a similar inequality for an arbitrary positive contraction α on the algebra and show that

$$(*) \qquad \tau(f(\alpha(a))) \leq \tau(\alpha(f(a))).$$

Here f is supposed to be a continuous convex function with $f(0) = 0$. (Note that $(*)$ holds without τ if f is operator convex; cf. Theorem IV.1 in [**1**].) The inequality $(*)$ supports the experience that inequalities involving operators are more simply treatable provided both sides are inside of a trace (see [**3** and **8**] for further evidences).

The idea of our proof is to approximate the operators a and $\alpha(a)$ by diagonal ones by means of their spectral resolution and to apply the classical Jensen's inequality in

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its discrete form. Comparing with [**3**] this way is more direct. (The main technical tool in [**3**] is the spectral dominance which is intimately related to the generalized s-numbers. See [**3** Remark 5; **5**, and **8**].)

Our basic references on operator algebras are [**4** and **10**]. We recall that if τ is a normal semifinite trace on a von Neumann algebra M, then τ is weak* lower semicontinuous and admits an extension to a linear functional on the ideal linearly spanned by the set $\{a \in M_+ : \tau(a) < +\infty\}$ (see [**4**, Chapter I, §6]. Let h be a selfadjoint operator with Jordan decomposition $h_1 - h_2$. Following [**3**] we say that $\tau(h)$ is defined if $\tau(h_1) < +\infty$ or $\tau(h_2) < +\infty$. In this case, we set $\tau(h) = \tau(h_1) - \tau(h_2)$. If both $\tau(h)$ and $\tau(h')$ are defined and $\tau(h) + \tau(h')$ is well defined (i.e., does not have the form $\infty - \infty$), then $\tau(h + h') = \tau(h) + \tau(h')$ (cf. Lemma 9 in [**3**]).

THEOREM A. *Let M and N be von Neumann algebras and $f: \mathbf{R}^+ \to \mathbf{R}$ be a continuous convex function with $f(0) = 0$. Assume that τ is a normal semifinite trace on M and $\alpha: N \to M$ is a positive contraction. Then for every $a \in N_+$ the inequality*

$$(*) \qquad \tau(f(\alpha(a))) \le \tau(\alpha(f(a)))$$

holds provided that both sides are defined.

PROOF. First we show that

$$(1) \qquad \tau(pf(p\alpha(a)p)p) \le \tau(p\alpha(f(a))p)$$

for any projection p in M with $\tau(p) < +\infty$ and for any $a \in N_+$. (Note that in this case both sides of (1) are defined and finite.) Take an arbitrary $\varepsilon > 0$. We approximate $p\alpha(a)p$ by a diagonal operator $\sum_{i=1}^n \mu_i p_i$ such that p_i is a spectral projection of $p\alpha(a)p$ for $1 \le i \le n$, $\sum_{i=1}^n p_i = p$,

$$\tau(pf(p\alpha(a)p)p) \le \sum_{i=1}^n f(\mu_i)\tau(p_i) + \varepsilon,$$

and

$$\mu_i = \begin{pmatrix} \tau(\alpha(a)p_i)/\tau(p_i) & \text{when } \tau(p_i) > 0, \\ 0 & \text{otherwise.} \end{pmatrix}$$

We choose $\delta > 0$ such that $|t_1 - t_2| \le \delta$ and $0 \le t_1 \le \|a\|$ imply $|f(t_1) - f(t_2)| \le \varepsilon/\tau(p)$. There exists a diagonal operator $b = \sum_{j=1}^k \nu_j q_j$ such that $\|a - b\| \le \delta$, $\sum_{j=1}^k q_j = I$, and

$$|\tau(p\alpha(f(a))p) - \tau(p\alpha(f(b))p)| \le \varepsilon.$$

By simple calculation we have

$$\begin{aligned} f(\mu_i) &= f\left(\sum_{j=1}^k \tau(\alpha(aq_j)p_i)/\tau(p_i)\right) \\ &= f\left(\sum_{j=1}^k \nu_j \tau(\alpha(q_j)p_i)/\tau(p_i) + \tau(\alpha(a-b)p_i)/\tau(p_i)\right) \end{aligned}$$

when $\tau(p_i) > 0$ and $1 \leq i \leq n$. Here

$$|\tau(\alpha(a-b)p_i)/\tau(p_i)| \leq \|\alpha(a-b)\| \leq \delta$$

and so

$$f(\mu_i) \leq f\left(\sum_{j=1}^{k} \nu_j \tau(\alpha(q_j)p_i)/\tau(p_i)\right) + \varepsilon/\tau(p).$$

Now apply Jensen's inequality

$$f(\mu_i) \leq \sum_{j=1}^{k} f(\nu_j)\tau(\alpha(q_j)p_i)/\tau(p_i) + \varepsilon/\tau(p).$$

Consequently

$$\begin{aligned}\sum_{i=1}^{n} f(\mu_i)\tau(p_i) &\leq \sum_{j=1}^{k} f(\nu_j)\tau\left(\alpha(q_j)\left(\sum_{i=1}^{n} p_i\right)\right) + \varepsilon \\ &= \tau(p\alpha(f(b))p) + \varepsilon.\end{aligned}$$

Finally we arrive at the estimate

$$\tau(pf(p\alpha(a)p)p) \leq \tau(p\alpha(f(a))p) + 3\varepsilon$$

and since ε was arbitrary (1) is proved.

When τ is finite the proof is complete. If not, then a supplementary argument is needed. There is $c \in \mathbf{R}^+ \cup \{+\infty\}$ such that $f \geq 0$ on $(c, +\infty) = I_1$ and $f \leq 0$ on $[0, c] = I_2$. Let q_i be the spectral projection of $\alpha(a)$ corresponding to I_i $(i = 1, 2)$. Since $q_1 f(\alpha(a)) - q_2(-f(\alpha(a)))$ is the Jordan decomposition of $f(\alpha(a))$ we have

$$\tau(f(\alpha(a))) = \tau(q_1 f(\alpha(a))) + \tau(q_2 f(\alpha(a))).$$

Let $d_1 - d_2$ be the Jordan decomposition of $\alpha(f(a))$. Since $\tau(d_1) < \infty$ or $\tau(d_2) < \infty$ the difference $\tau(pd_1) - \tau(pd_2)$ is well defined and equal to $\tau(p\alpha(f(a)))$ for any projection p.

By the semifiniteness of τ there exists an increasing net $(p_s^i)_{s \in S}$ of projections in M such that $p_s^i \to q_i$ $(i = 1, 2)$ and $\tau(Ip_s^i) < +\infty$ $(i = 1, 2$ and $s \in S)$. For any $s \in S$ we have proved that

$$\tau(p_s^i f(p_s^i \alpha(a) p_s^i) p_s^i) \leq \tau(p_s^i \alpha(f(a)) p_s^i).$$

Since τ is weak* lower semicontinuous and

$$p_s^i f(p_s^i \alpha(a) p_s^i) p_s^i \to q_i f(q_i \alpha(a) q_i) = q_i f(\alpha(a))$$

in the strong operator topology [**10**, II, 4.6] and hence in the weak* topology, we establish

$$\begin{aligned}\tau(q_i f(\alpha(a))) &\leq \liminf_s \tau(p_s^i f(p_s^i \alpha(a) p_s^i) p_s^i) \\ &\leq \lim_s \tau(p_s^i d_1) - \lim_s \tau(p_s^i d_2) = \tau(q_i \alpha(f(a))).\end{aligned}$$

Adding these inequalities for $i = 1$ and $i = 2$ we complete the proof.

In the whole proof of the theorem we needed $f(0) = 0$ only when the discrete Jensen's inequality was applied to a degenerate convex combination

$$\sum_{j=1}^{k} \nu_j \tau(\alpha(q_j)p_i)/\tau(p_i)$$

(that is, $\sum_{j=1}^{k} \tau(\alpha(q_j)p_i)/\tau(p_i) \leq 1$). If we replace $f(0) = 0$ by the assumption that α is unit preserving, then

$$\sum_{j=1}^{k} \tau(\alpha(q_j)p_i)/\tau(p_i) = \tau\left(\alpha\left(\sum_{j=1}^{k} q_j\right)p_i\right) \Big/ \tau(p_i) = 1$$

and the same arguments give another version of the previous theorem. (We recall that a positive unit preserving linear mapping is necessarily a contraction; see Lemma 1 in [**7**], for example.)

THEOREM B. *Let M and N be von Neumann algebras and $f: \mathbf{R}^+ \to \mathbf{R}$ be a continuous convex function with $f(0) = 0$. Assume that τ is a normal semifinite trace on M and $\alpha: N \to M$ is a positive unit preserving mapping. Then for every $a \in N_+$ the inequality*

$$(*) \qquad \tau(f(\alpha(a))) \leq \tau(\alpha(f(a)))$$

holds whenever both sides are defined.

Theorems A and B have a simple consequence for positive contractions on C^*-algebras. Recall that a positive functional τ on a C^*-algebra is called tracial if $\tau(aa^*) = \tau(a^*a)$ for every element a.

COROLLARY. *Let A and B be unitial C^*-algebras and $\alpha: A \to B$ be a positive contraction. Assume that $f: \mathbf{R}^+ \to \mathbf{R}$ is a continuous convex function. If τ is a tracial positive functional on B and*

(i) *α is unit preserving, or*

(ii) *$f(0) = 0$,*

then the inequality $(*)$ holds for every positive $a \in A$.

PROOF. The second adjoint of α is an extension of α and forms a positive contraction of A^{**} into B^{**}. On the other hand, τ admits an extension to a normal tracial positive functional of the enveloping von Neumann algebra B^{**}. So one simply applies Theorems A and B for the enveloping von Neumann algebras A^{**} and B^{**}.

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MATHEMATICS INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, REÁLTANODA U. 13-15, 1364 BUDAPEST, PF. 127, HUNGARY