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A COUNTEREXAMPLE TO A PROBLEM ON POINTS OF CONTINUITY IN BANACH SPACES

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ABSTRACT. In a previous paper of the first two authors [**GM**] the space JT_∞ was constructed as a James space over a tree with infinitely many branching points. It was proved that the predual B_∞ of JT_∞ fails the "point of continuity property."

In the present paper we show that B_∞ has the so-called "convex point of continuity property" thus answering a question of Edgar and Wheeler [**EW**] in the negative.

1. Definitions, notations and preliminaries. Recall [**BR**] that a Banach space X has the *point of continuity property* (PCP) if for every weakly closed bounded subset C of X there is $x \in C$ such that the weak and norm topology—restricted to C—coincide at x. An equivalent formulation goes as follows (compare [**B**, Proposition 1]): For every bounded subset C of X and for $\varepsilon > 0$ there is a relatively weakly open subset U of C such that the norm-diameter of U is less than ε.

Previously, J. Bourgain [**B**] introduced (under the name of property $(*)$) the following weaker concept: A Banach space X has the *convex point of continuity property* (CPCP) if every closed convex bounded subset C of X has a point x where the relative weak and norm topologies coincide. Again this may be rephrased as follows: For every *convex* bounded subset C of X and $\varepsilon > 0$ there is a relatively weakly open subset U of C of norm-diameter less than ε. The question whether PCP and CPCP are in fact equivalent remained open and was explicitly asked in [**EW**]. We shall show that the space B_∞ furnishes an example with CPCP but failing PCP.

Recall from [**GM**] the definition of JT_∞: We consider the tree with infinitely many branching points

$$T_\infty = \sum_{k=0}^{\infty} \mathbf{N}^k.$$

If $t = (n_1, \ldots, n_k) \in T_\infty$, set $|t| = k$ to be the *rank* of t. If $x = (x_t)_{t\in T_\infty}$ is a real-valued function of finite support defined on T_∞, let

$$||x||_{JT_\infty} = \sup \left(\sum_{i=1}^{n} \left(\sum_{t\in S_i} x_t \right)^2 \right)^{1/2},$$

the supremum taken over all families $(S_1, \ldots, S_n)$ of disjoint segments in T_∞. JT_∞ will be the completion of this normed space.

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As the finite vectors are dense in JT_∞, an element $y \in JT_\infty^*$ is characterized by its values $y_t = y(e_t)$ on the unit vectors e_t, $t \in T_\infty$. The space B_∞ will be the norm closure of the span of the coefficient functionals—again denoted $\{e_t, t \in T_\infty\}$—in JT_∞^*.

We shall say that a subset $A \subseteq T_\infty$ is full, if for each segment S, the intersection $S \cap A$ is again a segment. In this case, the projection

$$P_A: JT_\infty \to JT_\infty, \qquad \sum_{t \in T_\infty} x_t e_t \to \sum_{t \in A} x_t e_t$$

is a contraction. The adjoint of P_A—still denoted P_A—defines a contraction from B_∞ to B_∞.

We shall need the following cases of full sets $A \subseteq T_\infty$: If γ is a branch in T_∞, we denote by P_γ the projection defined by the set γ. L_n will denote the nth line of T_∞, i.e. L_n: $\{t\colon |t| = n\}$ and P_n the projection defined by L_n. If $n \le m$ we denote P_n^m the projection defined by $L_n \cup L_{n+1} \cup \cdots \cup L_m$.

For each branch $\gamma = \{\phi, (n_1), (n_1, n_2), \ldots, (n_1, n_2, \ldots, n_k), \ldots\}$ in T_∞, define

$$S_\gamma\colon\ JT_\infty^* \to \mathbf{R}, \qquad y \to \lim_{t \in \gamma} y_t$$

which is well defined as $P_\gamma(JT_\infty)$ is isometric to the usual James space J. The collection of branches γ may be identified—in an obvious way—with $\mathbf{N}^{\mathbf{N}}$. The following lemma is just a slight variant of Theorem 1 of [**LS**] and the proof carries over almost verbatim.

1.1 LEMMA. *The operator*

$$S\colon\ JT_\infty^* \to l^2(\mathbf{N}^{\mathbf{N}}), \qquad y \to \left(\lim_{t \in \gamma} y_t\right)_{\gamma \in \mathbf{N}^{\mathbf{N}}}$$

is a well-defined quotient map and the kernel of S equals B_∞. □

Note the following consequence of Lemma 1.1 which we shall use in the sequel:

1.2 LEMMA. *Let $y \in JT_\infty^*$ such that*

$$(*) \qquad \liminf_{n \to \infty} \|P_n(y)\|_\infty = 0,$$

where $\|\cdot\|_\infty$ denotes the sup-norm for a function defined on T_∞. Then y is an element of B_∞. In particular, if $(*)$ holds true, there cannot be an increasing sequence $(n_k)_{k=1}^\infty$ and $\alpha > 0$ such that

$$(**) \qquad \|P_{n_k+1}^{n_{k+1}}(y)\|_{JT_\infty^*} \ge \alpha, \qquad k = 1, 2, \ldots.$$

PROOF. Condition $(*)$ implies that for each branch γ the limit $S_\gamma(y)$ equals 0; hence $S(y) = 0$ and by Lemma 1.1 we infer that $y \in B_\infty$.

For the second part, note that B_∞ is spanned by the coefficient functionals e_t, $t \in T_\infty$, hence $(**)$ is contradictory to $(*)$. □

2. The main result. The following lemma is crucial for the proof of Theorem 2.2. The proof is an interplay of a gliding hump argument with the formation of Cesàro means.

2.1 LEMMA. *Let C be a convex, bounded subset of B_∞, $M \in \mathbf{N}$, and $\varepsilon > 0$. There is a relatively weakly open convex subset U of C and $N > M$ such that*

$$\|P_N(U)\|_\infty = \sup\{\|P_N(y)\|_\infty\colon\ y \in U\} \leq \varepsilon.$$

PROOF. We may and do suppose that C is contained in the unit ball of B_∞ and that $0 < \varepsilon < 1$. Choose natural numbers n and m such that $n > 2/\varepsilon$ and $m > (8n/\varepsilon^2) + 1$ and choose $\eta > 0$ such that $\eta < \varepsilon/2mn$. Fix an element y_0^1 in C and choose $N > M$ such that $\|P_N(y_0^1)\|_\infty < \eta$. Suppose the lemma is not true (for the N chosen above). We shall obtain a contradiction by double-induction: For $0 \leq i \leq n$ and $1 \leq j \leq m$ we shall find $y_i^j \in C$ and, for $1 \leq i \leq n$ and $1 \leq j \leq m$, $t_i^j \in L_N$ such that

$$
\begin{array}{lll}
\text{(i)} & |y_i^j(t_i^j)| > \varepsilon, & 1 \leq j \leq m,\ 1 \leq i \leq n, \\
\text{(ii)} & y_0^j = n^{-1}(y_1^{j-1} + \cdots + y_n^{j-1}), & 2 \leq j \leq m, \\
\text{(iii)} & |(y_i^j - y_0^j)(t_p^q)| < \eta, & \text{if } (q,p) < (j,i) \text{ lexicographically,} \\
 & & \text{i.e. } q < j \text{ or } q = j \text{ and } p < i, \\
\text{(iv)} & |y_i^j(t_p^q)| < \eta & \text{if } (q,p) > (j,i) \text{ lexicographically.}
\end{array}
$$

Let us suppose for the moment that we have done the construction and finish the proof. Fix (q,p) with $1 \leq q < m$ and $1 \leq p \leq n$. Note that (ii) and (iv) imply $|y_0^q(t_p^q)| < \eta$. Now apply (i) (for (j,i) equal to (q,p)), (ii), (iii), and (iv) to obtain

$$|y_0^{q+1}(t_p^q)| > \varepsilon/n - 2\eta.$$

Repeated application of (ii) and (iii) shows

$$|y_0^m(t_p^q)| > \varepsilon/n - (m-q+1)\eta \geq \varepsilon/n - m\eta \geq \varepsilon/2n.$$

Hence

$$\|y_0^m\|_{B_\infty}^2 \geq \|P_N(y_0^m)\|_{B_\infty}^2 \geq \sum_{q=1}^{m-1}\sum_{p=1}^{n} |y_0^m(t_p^q)|^2$$
$$\geq (m-1)\cdot n \cdot (\varepsilon/2n)^2 \geq 2$$

which contradicts the assumption that C is contained in the unit ball of B_∞.

So, let us do the inductive construction. We have already chosen y_0^1; let $C_0^1 = C$. By assumption there is $y_1^1 \in C_0^1$ and $t_1^1 \in L_N$ s.t. $|y_1^1(t_1^1)| > \varepsilon$. Let

$$A_1^1 = \{t \in L_N\colon\ |y_1^1(t)| > \eta\}$$

and

$$C_1^1 = \{y \in C_0^1\colon\ |(y - y_0^1)(t)| < \eta \text{ for } t \in A_1^1\}.$$

Clearly C_1^1 is a relatively weakly open convex subset of C; hence by assumption there is $y_2^1 \in C_1^1$ and $t_2^1 \in L_N$ s.t. $|y_2^1(t_2^1)| > \varepsilon$. Let

$$A_2^1 = A_1^1 \cup \{t \in L_N\colon\ |y_2^1(t)| > \eta\}$$

and

$$C_2^1 = \{y \in C_0^1\colon\ |(y - y_0^1)(t)| < \eta \text{ for } t \in A_2^1\}.$$

Continue in an obvious way to find $y_1^1, \ldots, y_n^1$ and $t_1^1, \ldots, t_n^1$ satisfying (i), (iii), and (iv). Define

$$y_0^2 = n^{-1}(y_1^1 + \cdots + y_n^1), \qquad A_0^2 = A_n^1,$$

and

$$C_0^2 = \{y \in C_0^1\colon\ |(y - y_0^2)(t)| < \eta \text{ for } t \in A_0^2\}.$$

Again C_0^2 is a relatively weakly open, nonempty, and convex subset of C; hence we may find $y_1^2 \in C_0^2$ and $t_1^2 \in L_N$ such that $|y_1^2(t_1^2)| > \varepsilon$.

Note that t_1^2 cannot belong to A_0^2 as the elements of C_0^2 are smaller than $2\eta + n^{-1}$ on the elements of A_0^2. Hence (iv) is satisfied for $(q,p) = (2,1)$ and $(i,j) < (q,p)$. Let

$$A_1^2 = A_0^2 \cup \{t \in L_N\colon\ |y_1^2(t)| > \eta\}$$

and

$$C_1^2 = \{y \in C_0^2\colon\ |(y - y_0^2)(t)| < \eta \text{ for } t \in A_1^2\}.$$

Again by assumption there is $y_2^2 \in C_1^2$ and $t_2^2 \in L_N$ s.t. $|y_2^2(t_2^2)| > \varepsilon$, etc.; we thus find $y_1^2, \dots, y_n^2$ and $t_1^2, \dots, t_n^2$ satisfying (i), (iii), and (iv). Now define

$$y_0^3 = n^{-1}(y_1^2 + \cdots + y_n^2), \qquad A_0^3 = A_n^2,$$
$$C_0^3 = \{y \in C_0^2\colon\ |(y - y_0^3)(t)| < \eta \text{ for } t \in A_0^3\}.$$

Continue in an obvious way for $j = 1, \dots, m$ to finish the inductive procedure and the proof of the lemma. □

2.2 THEOREM. *B_∞ has CPCP.*

PROOF. If the theorem were false we could find a convex bounded $C \subseteq B_\infty$ and $\alpha > 0$ such that each relatively weakly open subset U of C has norm-diameter greater than 3α.

Again we shall argue inductively: Let $y_1 \in C$, $||y_1|| > \alpha$ and find $x_1 \in JT_\infty$, $||x_1|| = 1$ of finite support, say $\operatorname{supp}(x_1) \subseteq L_1 \cup \cdots \cup L_{M_1}$ s.t. $\langle x_1, y_1 \rangle > \alpha$. Let

$$C_1 = \{y \in C\colon\ \langle x_1, y \rangle > \alpha\}$$

and apply Lemma 2.1 to C_1 to find $N_1 > M_1$ and a relatively weakly open, nonempty, and convex $D_1 \subseteq C_1$ s.t. $||P_N(D_1)||_\infty < 1$. Finally note that $P_1^{N_1}(B_\infty)$ is isomorphic to l^2, which has CPCP, hence we may find a relatively weakly open, nonempty, convex $E_1 \subseteq D_1$ s.t.

$$\operatorname{diam}(P_1^{N_1}(E_1)) < \alpha;$$

hence

$$\operatorname{diam}(P_{N_1+1}^{\infty}(E_1)) > 2\alpha.$$

Find $y_2 \in E_1$ and a finitely valued $x_2 \in JT_\infty$ with the support contained in $L_{N_1+1} \cup \cdots \cup L_{M_2}$, $||x_2|| = 1$, s.t. $\langle x_2, y_2 \rangle > \alpha$ and let

$$C_2 = \{y \in E_1\colon \langle x_2, y \rangle > \alpha\}.$$

Apply Lemma 2.1 to find $N_2 > M_2$ and $D_2 \subseteq C_2$, s.t. $||P_{N_2}(D_2)||_\infty < 1/2$. Finally find $E_2 \subseteq D_2$ s.t. $\operatorname{diam}(P_1^{N_2}(E_2)) < \alpha$; hence $\operatorname{diam}(P_{N_2+1}^{\infty}(E_2)) > 2\alpha$.

Continue in an obvious way to find $C_n \supset D_n \supset E_n \supset C_{n+1} \supset \cdots$, $M_n < N_n < M_{n+1} < N_{n+1} < \cdots$, and $x_n \in JT_\infty$, $||x_n|| = 1$ with $\operatorname{supp}(x_n) \subseteq L_{N_{n-1}+1} \cup \cdots \cup L_{M_n}$ s.t.

$$C_m \subseteq \{y \in B_\infty\colon\ \langle x_n, y \rangle > \alpha\}, \qquad m > n,$$

while

$$||P_{N_n}(C_m)||_\infty < n^{-1}, \qquad m > n.$$

Let $\overline{C}_n^*$ be the $\sigma(JT_\infty^*, JT_\infty)$-closure of C_n. By weak-star compactness there is an element $y \in JT_\infty^*$,

$$y \in \bigcap_{n=1}^{\infty} \overline{C}_n^*.$$

This y has the (impossible) properties described in Lemma 1.2, so we arrive at a contradiction and prove the theorem. □

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