A COUNTEREXAMPLE TO A PROBLEM ON POINTS OF CONTINUITY IN BANACH SPACES
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ABSTRACT. In a previous paper of the first two authors [GM] the space $J^\infty_\infty$ was constructed as a James space over a tree with infinitely many branching points. It was proved that the predual $B_\infty$ of $J^\infty_\infty$ fails the "point of continuity property."

In the present paper we show that $B_\infty$ has the so-called "convex point of continuity property" thus answering a question of Edgar and Wheeler [EW] in the negative.

1. Definitions, notations and preliminaries. Recall [BR] that a Banach space $X$ has the point of continuity property (PCP) if for every weakly closed bounded subset $C$ of $X$ there is $x \in C$ such that the weak and norm topology—restricted to $C$—coincide at $x$. An equivalent formulation goes as follows (compare [B, Proposition 1]): For every bounded subset $C$ of $X$ and for $\varepsilon > 0$ there is a relatively weakly open subset $U$ of $C$ such that the norm-diameter of $U$ is less than $\varepsilon$.

Previously, J. Bourgain [B] introduced (under the name of property (*)) the following weaker concept: A Banach space $X$ has the convex point of continuity property (CPCP) if every closed convex bounded subset $C$ of $X$ has a point $x$ where the relative weak and norm topologies coincide. Again this may be rephrased as follows: For every convex bounded subset $C$ of $X$ and $\varepsilon > 0$ there is a relatively weakly open subset $U$ of $C$ of norm-diameter less than $\varepsilon$. The question whether PCP and CPCP are in fact equivalent remained open and was explicitly asked in [EW]. We shall show that the space $B_\infty$ furnishes an example with CPCP but failing PCP.

Recall from [GM] the definition of $J^\infty_\infty$: We consider the tree with infinitely many branching points

$$T_\infty = \sum_{k=0}^{\infty} \mathbb{N}^k.$$ 

If $t = (n_1, \ldots, n_k) \in T_\infty$, set $|t| = k$ to be the rank of $t$. If $x = (x_t)_{t \in T_\infty}$ is a real-valued function of finite support defined on $T_\infty$, let

$$||x||_{J^\infty_\infty} = \sup \left( \sum_{i=1}^{n} \left( \sum_{t \in S_i} x_t \right)^2 \right)^{1/2},$$

the supremum taken over all families $(S_1, \ldots, S_n)$ of disjoint segments in $T_\infty$. $J^\infty_\infty$ will be the completion of this normed space.
As the finite vectors are dense in $JT_\infty$, an element $y \in JT_\infty^*$ is characterized by its values $y_t = y(e_t)$ on the unit vectors $e_t$, $t \in T_\infty$. The space $B_\infty$ will be the norm closure of the span of the coefficient functionals—again denoted $\{e_t, t \in T_\infty\}$—in $JT_\infty^*$.

We shall say that a subset $A \subseteq T_\infty$ is full, if for each segment $S$, the intersection $S \cap A$ is again a segment. In this case, the projection

$$P_A: JT_\infty \to JT_\infty,$$

$$\sum_{t \in T_\infty} x_t e_t \to \sum_{t \in A} x_t e_t$$

is a contraction. The adjoint of $P_A$—still denoted $P_A$—defines a contraction from $B_\infty$ to $B_\infty$.

We shall need the following cases of full sets $A \subseteq T_\infty$: If $\gamma$ is a branch in $T_\infty$, we denote by $P_\gamma$ the projection defined by the set $\gamma$. $L_n$ will denote the $n$th line of $T_\infty$, i.e. $L_n: \{t: |t| = n\}$ and $P_n$ the projection defined by $L_n$. If $n \leq m$ we denote $P_n^m$ the projection defined by $L_n \cup L_{n+1} \cup \cdots \cup L_m$.

For each branch $\gamma = \{\phi, (n_1), (n_2), \ldots, (n_1, n_2, \ldots, n_k), \ldots\}$ in $T_\infty$, define

$$S_\gamma: JT_\infty^* \to \mathbb{R}, \quad y \to \lim_{t \in \gamma} y_t$$

which is well defined as $P_\gamma(JT_\infty)$ is isometric to the usual James space $J$. The collection of branches $\gamma$ may be identified—in an obvious way—with $\mathbb{N}^\infty$. The following lemma is just a slight variant of Theorem 1 of [LS] and the proof carries over almost verbatim.

1.1 LEMMA. The operator

$$S: JT_\infty^* \to l^2(\mathbb{N}^\infty), \quad y \to \left(\lim_{t \in \gamma} y_t\right)_{\gamma \in \mathbb{N}^\infty}$$

is a well-defined quotient map and the kernel of $S$ equals $B_\infty$. □

Note the following consequence of Lemma 1.1 which we shall use in the sequel:

1.2 LEMMA. Let $y \in JT_\infty^*$ such that

$$(*) \quad \lim_{n \to \infty} \inf \|P_n(y)\|_\infty = 0,$$

where $\| \cdot \|_\infty$ denotes the sup-norm for a function defined on $T_\infty$. Then $y$ is an element of $B_\infty$. In particular, if $(*)$ holds true, there cannot be an increasing sequence $(n_k)_{k=1}^\infty$ and $\alpha > 0$ such that

$$(**) \quad \|P_{n_k+1}^k(y)\|_{JT_\infty^*} \geq \alpha, \quad k = 1, 2, \ldots.$$

PROOF. Condition $(*)$ implies that for each branch $\gamma$ the limit $S_\gamma(y)$ equals 0; hence $S(y) = 0$ and by Lemma 1.1 we infer that $y \in B_\infty$.

For the second part, note that $B_\infty$ is spanned by the coefficient functionals $e_t$, $t \in T_\infty$, hence $(**)$ is contradictory to $(*)$. □

2. The main result. The following lemma is crucial for the proof of Theorem 2.2. The proof is an interplay of a gliding hump argument with the formation of Cesàro means.
2.1 Lemma. Let $C$ be a convex, bounded subset of $B_\infty$, $M \in \mathbb{N}$, and $\varepsilon > 0$. There is a relatively weakly open convex subset $U$ of $C$ and $N > M$ such that

$$||P_N(U)||_\infty = \sup\{||P_N(y)||_\infty : y \in U\} \leq \varepsilon.$$

Proof. We may and do suppose that $C$ is contained in the unit ball of $B_\infty$ and that $0 < \varepsilon < 1$. Choose natural numbers $n$ and $m$ such that $n > 2/\varepsilon$ and $m > (8n/\varepsilon^2) + 1$ and choose $\eta > 0$ such that $\eta < \varepsilon/2mn$. Fix an element $y_0^1$ in $C$ and choose $N > M$ such that $||P_N(y_0^1)||_\infty < \eta$. Suppose the lemma is not true (for the $N$ chosen above). We shall obtain a contradiction by double-induction: For $0 \leq i \leq n$ and $1 \leq j \leq m$ we shall find $y_i^j \in C$ and, for $1 \leq i \leq n$ and $1 \leq j \leq m$, $t_i^j \in L_N$ such that

1. $|y_i^j(t_i^j)| > \varepsilon$,
2. $y_0^0 = n^{-1}(y_1^1 + \cdots + y_n^1)$,
3. $|(y_i^j - y_0^0)(t_p^j)| < \eta$, if $(q,p) < (j,i)$ lexicographically,
4. $|y_i^j(t_p^j)| < \eta$, if $(q,p) > (j,i)$ lexicographically.

Let us suppose for the moment that we have done the construction and finish the proof. Fix $(q,p)$ with $1 \leq q < m$ and $1 \leq p \leq n$. Note that (ii) and (iv) imply $|y_0^0(t_p^j)| < \eta$. Now apply (i) (for $(j,i)$ equal to $(q,p)$), (ii), (iii), and (iv) to obtain

$$|y_0^{q+1}(t_p^j)| > \varepsilon/n - 2\eta.$$ 

Repeated application of (ii) and (iii) shows

$$|y_0^m(t_p^j)| > \varepsilon/n - (m - q + 1)n \geq \varepsilon/n - mn \geq \varepsilon/2n.$$ 

Hence

$$||y_0^m||_{B_\infty}^2 \geq ||P_N(y_0^m)||_{B_\infty}^2 \geq \sum_{q=1}^{m-1} \sum_{p=1}^n |y_0^m(t_p^j)|^2 \geq (m - 1) \cdot n \cdot (\varepsilon/2n)^2 \geq 2$$

which contradicts the assumption that $C$ is contained in the unit ball of $B_\infty$.

So, let us do the inductive construction. We have already chosen $y_0^1$; let $C_0^1 = C$. By assumption there is $y_1^1 \in C_0^1$ and $t_1^1 \in L_N$ s.t. $|y_1^1(t_1^1)| > \varepsilon$. Let

$$A_1^1 = \{t \in L_N: |y_1^1(t)| > \eta\}$$

and

$$C_1^1 = \{y \in C_0^1: |(y - y_0^1)(t)| < \eta \text{ for } t \in A_1^1\}.$$ 

Clearly $C_1^1$ is a relatively weakly open convex subset of $C$; hence by assumption there is $y_2^1 \in C_1^1$ and $t_2^1 \in L_N$ s.t. $|y_2^1(t_2^1)| > \varepsilon$. Let

$$A_2^1 = A_1^1 \cup \{t \in L_N: |y_2^1(t)| > \eta\}$$

and

$$C_2^1 = \{y \in C_0^1: |(y - y_0^1)(t)| < \eta \text{ for } t \in A_2^1\}.$$ 

Continue in an obvious way to find $y_1^1, \ldots, y_n^1$ and $t_1^1, \ldots, t_n^1$ satisfying (i), (iii), and (iv). Define

$$y_0^2 = n^{-1}(y_1^1 + \cdots + y_n^1), \quad A_0^2 = A_1^1,$$
and

$$C_0^2 = \{ y \in C_0^1: |(y - y_0^2)(t)| < \eta \text{ for } t \in A_0^2 \}.$$ 

Again $C_0^2$ is a relatively weakly open, nonempty, and convex subset of $C$; hence we may find $y_0^2 \in C_0^2$ and $t_0^2 \in L_N$ such that $|y_0^2(t_0^2)| > \varepsilon$.

Let $t_0^2$ cannot belong to $A_0^2$ as the elements of $C_0^2$ are smaller than $2\eta + n^{-1}$ on the elements of $A_0^2$. Hence (iv) is satisfied for $(q,p) = (2,1)$ and $(i,j) < (q,p)$. Let

$$A_2^2 = A_0^2 \cup \{ t \in L_N: |y_0^2(t)| > \eta \}$$

and

$$C_0^3 = \{ y \in C_0^2: |(y - y_0^3)(t)| < \eta \text{ for } t \in A_0^3 \}.$$ 

Again by assumption there is $y_0^3 \in C_0^3$ and $t_0^3 \in L_N$ s.t. $|y_0^3(t_0^3)| > \varepsilon$, etc.; we thus find $y_0^1, \ldots, y_0^2$ and $t_0^1, \ldots, t_0^2$ satisfying (i), (iii), and (iv). Now define

$$y_0^3 = n^{-1}(y_0^1 + \cdots + y_0^2), \quad A_0^3 = A_0^2,$$

$$C_0^3 = \{ y \in C_0^2: |(y - y_0^3)(t)| < \eta \text{ for } t \in A_0^3 \}.$$ 

Continue in an obvious way for $j = 1, \ldots, m$ to finish the inductive procedure and the proof of the lemma. □

2.2 Theorem. $B_\infty$ has CPCP.

Proof. If the theorem were false we could find a convex bounded $C \subseteq B_\infty$ and $\alpha > 0$ such that each relatively weakly open subset $U$ of $C$ has norm-diameter greater than $3\alpha$.

Again we shall argue inductively: Let $y_1 \in C$, $||y_1|| > \alpha$ and find $x_1 \in JT_\infty$, $||x_1|| = 1$ of finite support, say $\text{supp}(x_1) \subseteq L_1 \subseteq \cdots \subseteq L_m$, s.t. $\langle x_1, y_1 \rangle > \alpha$. Let

$$C_1 = \{ y \in C: \langle x_1, y \rangle > \alpha \}$$

and apply Lemma 2.1 to $C_1$ to find $N_1 > M_1$ and a relatively weakly open, nonempty, and convex $D_1 \subseteq C_1$ s.t. $||P_{N_1}(D_1)||_\infty < 1$. Finally note that $P_{N_1}^{N_1}(B_\infty)$ is isomorphic to $l^2$, which has CPCP; hence we may find a relatively weakly open, nonempty, convex $E_1 \subseteq D_1$ s.t.

$$\text{diam}(P_{N_1}^{N_1}(E_1)) < \alpha;$$

hence

$$\text{diam}(P_{N_1+1}^{N_1}(E_1)) > 2\alpha.$$ 

Find $y_2 \in E_1$ and a finitely valued $x_2 \in JT_\infty$ with the support contained in $L_{N_1+1} \subseteq \cdots \subseteq L_{M_2}$, $||x_2|| = 1$, s.t. $\langle x_2, y_2 \rangle > \alpha$ and let

$$C_2 = \{ y \in E_1: \langle x_2, y \rangle > \alpha \}.$$ 

Apply Lemma 2.1 to find $N_2 > M_2$ and $D_2 \subseteq C_2$, s.t. $||P_{N_2}(D_2)||_\infty < 1/2$. Finally find $E_2 \subseteq D_2$ s.t. $\text{diam}(P_{N_2}^{N_2}(E_2)) < \alpha$; hence $\text{diam}(P_{N_2+1}^{N_2}(E_2)) > 2\alpha$.

Continue in an obvious way to find $C_n \supset D_n \supset E_n \supset C_{n+1} \supset \cdots$, $M_n < N_n < M_{n+1} < N_{n+1} < \cdots$, and $x_n \in JT_\infty$, $||x_n|| = 1$ with $\text{supp}(x_n) \subseteq L_{N_{n-1}+1} \subseteq \cdots \subseteq L_{M_n}$ s.t.

$$C_m \subseteq \{ y \in B_\infty: \langle x_n, y \rangle > \alpha \}, \quad m > n,$$

while

$$||P_{N_m}(C_m)||_\infty < n^{-1}, \quad m > n.$$
Let $\overline{C}_n^*$ be the $\sigma(JT^*_\infty, JT_{\infty})$-closure of $C_n$. By weak-star compactness there is an element $y \in JT^*_\infty$,

$$y \in \bigcap_{n=1}^{\infty} \overline{C}_n^*.$$ 

This $y$ has the (impossible) properties described in Lemma 1.2, so we arrive at a contradiction and prove the theorem. □

REFERENCES


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