ITERATIVE APPROXIMATION OF FIXED POINTS OF LIPSCHITZIAN STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. Suppose $X = L_p$ (or $l_p$), $p > 2$, and $K$ is a nonempty closed convex bounded subset of $X$. Suppose $T: K \rightarrow K$ is a Lipschitzian strictly pseudo-contractive mapping of $K$ into itself. Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying:

(i) $0 < C_n < 1$ for all $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} C_n = \infty$, and
(iii) $\sum_{n=1}^{\infty} C_n^2 < \infty$.

Then the iteration process, $x_0 \in K,$

$$x_{n+1} = (1 - C_n)x_n + C_nTx_n$$

for $n \geq 1$, converges strongly to a fixed point of $T$ in $K$.

1. Introduction. Let $X$ be a Banach space, $K \subseteq X$. A mapping $T: K \rightarrow K$ is said to be a **strict pseudo-contraction** if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all $x, y$ in $K$ and $r > 0$. If, in the above definition, $t = 1$, then $T$ is said to be a **pseudo-contraction**. Pseudo-contraction mappings have been studied by various authors (see e.g., [1, 5, 6, 10, 11, 12]). Interest in such mappings stems mainly from the fact that a mapping $T$ is pseudo-contraction if and only if $(I - T)$ is accretive [5, Proposition 1], where, for a mapping $U$ with domain $D(U)$ and range $R(U)$ in an arbitrary Banach space $X$, $U$ is said to be accretive [5] if the inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

holds for every $x$ and $y$ in $D(U)$ and for **all** $s > 0$. If (2) holds only for some $s > 0$, $U$ is said to be **monotone** [12]. The mapping theory for accretive mappings is thus closely related to the fixed point theory of pseudo-contractive mappings.

The accretive operators were introduced independently by T. Kato [12] and F. E. Browder [5] in 1967. An early fundamental result in the theory of accretive operators, due to Browder [5], states that the initial value problem

$$du/dt + Tu = 0, \quad u(0) = \omega$$

is solvable if $T$ is locally Lipschitizian and accretive on $X$, a result which was subsequently generalized by R. H. Martin [16] to the continuous accretive operators. In [2], J. Bogin considered the connection between strict pseudo-contractions and strictly accretive operators (defined below). He proved that $U$ is a strict pseudo-contraction if and only if $(I - U)$ is a strict accretive operator. He further proved a

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fixed point theorem in Banach spaces for Lipschitz strict pseudo-contractions, and as a consequence obtained a mapping theorem of Browder for Lipschitzian strictly accretive operators.

Suppose now $X = L_p$ (or $l_p$, $p \geq 2$, and $K \subseteq X$. Suppose further that $T: K \to K$ is a Lipschitz strict pseudo-contraction with a nonempty fixed point set in $K$. Our objective in this paper is to prove that an iterative process of the type introduced by W. R. Mann [15] converges strongly to a fixed point of $T$.

REMARK. For Lipschitz pseudo-contractions with a nonempty fixed point set, it is an open question, even in Hilbert spaces, whether or not an iteration process of the Mann-type will converge strongly to a fixed point of $T$ (see [11, p. 504].

2. Preliminaries. For a Banach space $X$ we shall denote by $J$ the duality mapping from $X$ to $2X^*$ given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\},$$

where $X^*$ denotes the dual space of $X$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If $X^*$ is strictly convex, then $J$ is single-valued, and if $X^*$ is uniformly convex, then $J$ is uniformly continuous on bounded sets (see [18, 19]). Thus, by a single-valued normalized duality mapping, we shall mean a mapping $j: X \to X^*$ such that for each $u$ in $X$, $j(u)$ is an element of $X^*$ which satisfies the following two conditions:

$$\langle j(u), u \rangle = \|j(u)\| \cdot \|u\|, \quad \|j(u)\| = \|u\|.$$

The accretiveness (or monotonicity) for $U$ defined in (2) can also be expressed in terms of the duality map $J$ as follows (see [12]). For each $x, y \in D(U)$, there is some $j \in J(x - y)$ such that

$$Re(Ux - Uy, j) \geq 0$$

and (as was observed in [12]) if $X$ is a Hilbert space, (3) is equivalent to the monotonicity of $U$ in the sense of Minty [17].

Now let $K \subseteq X$. A mapping $A: K \to X$ is said to be strictly accretive if for each $x, y$ in $K$ there exists $\omega \in J(x - y)$ such that

$$(Ax - Ay, \omega) \geq k\|x - y\|^2$$

for some constant $k > 0$. Without loss of generality we shall assume $k \in (0, 1)$.

In the sequel we shall assume that $L_p$, $p \geq 2$, has at least two disjoint sets of positive finite measure, $X$ will denote $L_p$ or $l_p$ ($p \geq 2$), and $j$ will always denote the single-valued normalized duality mapping of $X$ into $X^*$. We shall need the following results.

**Lemma 1.** For the Banach space $X$, the following inequality holds for all $x, y$ in $X$:

$$\|x + y\|^2 \leq (p - 1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y) \rangle.$$  

**Proof.** For $X = L_p$ or $l_p$, $p \geq 2$, the following inequality holds (see, e.g., [7]). For all $x, y \in X$,

$$(p - 1)\|x + y\|^2 \geq \|x\|^2 + \|y\|^2 + 2\langle y, j(x) \rangle.$$  

Now, replace $x$ by $y$ and $y$ by $x - y$ to get

$$\|x - y\|^2 \leq (p - 1)\|x\|^2 + \|y\|^2 + 2\langle -x, j(y) \rangle.$$  

Now replace $x$ by $-x$ to obtain (5).
Lemma 2 (Dunn [9, p. 41]). Let $\beta_n$ be recursively generated by
\[ \beta_{n+1} = (1 - \delta_n)\beta_n + \sigma_n^2 \]
with $n \geq 1$, $\beta_1 \geq 0$, $\{\delta_n\} \subseteq [0,1]$, and
\begin{align*}
(7a) \quad & \sum_{n=1}^{\infty} \sigma_n^2 < \infty, \\
(7b) \quad & \sum_{n=1}^{\infty} \delta_n = \infty.
\end{align*}
Then $\beta_n \geq 0$, for $n \geq 1$, and $\beta_n \to 0$ as $n \to \infty$.

Lemma 3 (Bogin [2]). Let $X$ be a Banach space, $K$ a subset of $X$, and $U: K \to X$. Then, if $U$ is a strict pseudo-contraction, $T = I - U$ is strictly monotone, with $k = (t - 1)/t$.

Proof. $U$ is a strict contraction implies that for all $x, y \in K$, and $r > 0$, $t > 0$, we have
\[ \|x - y\| \geq \|(1 + r)(x - y) - rt(Ux - Uy)\| \]
\[ = \|(1 + r)(x - y) - r(tUx - tUy)\|. \]
Thus, the mapping $(tU)$ is pseudo-contractive, so by [5], the mapping $T_t$ defined by $T_t = I - (tU)$ is monotone. So, for each $x, y$ in $K$ there exists $j \in J(x - y)$ such that
\[ \langle T_t x - T_t y, j(x - y) \rangle \geq 0. \]
Observe that $T_t = I - (tU) = I - t(I - T) = tT - (t - 1)I$, so that the above inequality yields
\[ t\langle Tx - Ty, j(x - y) \rangle - (t - 1)\langle x - y, j(x - y) \rangle \geq 0 \]
which simplifies to
\[ \langle Tx - Ty, j(x - y) \rangle \geq \frac{(t - 1)}{t} \langle x - y, j(x - y) \rangle = k\|x - y\|^2, \]
where $k = (t - 1)/t$, establishing the lemma.

3. Main result.

Theorem. Suppose $K$ is a nonempty closed bounded convex subset of $X$ and $T: K \to K$ is a Lipschitz strictly pseudo-contractive mapping of $K$ into itself. Let $\{C_n\}$ be a real sequence satisfying:
(i) $0 < C_n < 1$ for all $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} C_n = \infty$,
(iii) $\sum_{n=1}^{\infty} C_n^2 < \infty$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by $x_1 \in K$,
\begin{align*}
(8) \quad & \quad x_{n+1} = (1 - C_n)x_n + C_nTx_n,
\end{align*}
converges strongly to a fixed point of $T$.

Proof. The existence of a fixed point follows from Deimling [8].
Let \( p \) be a fixed point of \( T \). Since \( T \) is strictly pseudo-contractive, then \((I - T)\) is strictly accretive. Thus, there exists some \( k \in (0,1) \) such that for each \( x, y \) in \( K \),
\[
\text{Re}((I - T)x - (I - T)y, j(x - y)) \geq k\|x - y\|^2.
\]
In particular,
\[
(9) \quad \text{Re}((I - T)x - (I - T)p, j(x - p)) \geq k\|x - p\|^2.
\]
From (8),
\[
\|x_{n+1} - p\|^2 = \|(1 - C_n)(x_n - p) + C_n(Tx_n - Tp)\|^2
\]
\[
= (1 - C_n)^2\|(x_n - p) + C_n(1 - C_n)^{-1}(Tx_n - Tp)\|^2
\]
\[
\leq (1 - C_n)^2\|x_n - p\|^2 + C_n^2(1 - C_n)^{-2}(p - 1)^2\|Tx_n - Tp\|^2
\]
\[
+ 2C_n(1 - C_n)^{-1}(j(Tx_n - Tp), j(x_n - p))
\]
so that
\[
\|x_{n+1} - p\|^2 \leq (1 - C_n)^2\|x_n - p\|^2 + C_n^2(p - 1)^2\|x_n - p\|^2
\]
\[
- 2C_n(1 - C_n)(Tx_n - Tp, j(x_n - p))
\]
\[
= (1 - C_n)^2\|x_n - p\|^2 + (p - 1)C_n^2L^2\|x_n - p\|^2
\]
\[
- 2C_n(1 - C_n)(Tx_n - Tp, j(x_n - p))
\]
\[
- 2C_n(1 - C_n)(x_n - p, j(x_n - p))
\]
\[
+ 2C_n(1 - C_n)(x_n - p, j(x_n - p))
\]
\[
= (1 - C_n)^2\|x_n - p\|^2 + (p - 1)L^2C_n^2\|x_n - p\|^2
\]
\[
+ 2C_n(1 - C_n)\|x_n - p\|^2
\]
\[
- 2C_n(1 - C_n)(x_n - Tx_n - p + Tp, j(x_n - p))
\]
\[
= [(1 - C_n)^2 + 2C_n(1 - C_n)]\|x_n - p\|^2 + (p - 1)L^2C_n^2\|x_n - p\|^2
\]
\[
- 2C_n(1 - C_n)((I - T)x_n - (I - T)p, j(x_n - p))
\]
\[
\leq [(1 - C_n)^2 + 2(1 - k)C_n(1 - C_n)]\|x_n - p\|^2
\]
\[
+ (p - 1)L^2C_n^2\|x_n - p\|^2
\]
\[
\leq [(1 - C_n)^2 + 2(1 - k)C_n(1 - C_n)]\|x_n - p\|^2 + d^2C_n^2,
\]
where
\[
d = (p - 1)^{1/2}L \sup_{n \geq 1} \|x_n - p\|
\]
and clearly, by adding \((1 - k)^2C_n^2\|x_n - p\|^2\) to the right side of the above inequality, we obtain
\[
\|x_{n+1} - p\|^2 \leq [(1 - C_n)^2 + 2(1 - k)C_n(1 - C_n) + (1 - k)^2C_n^2]\|x_n - p\|^2 + d^2C_n^2
\]
\[
= [1 - (1 - k)C_n]^2\|x_n - p\|^2 + d^2C_n^2.
\]
Set \( \rho_n = \|x_n - p\|^2, 1 - \gamma_n = [1 - (1 - k)C_n]^2 \geq 0 \) to obtain
\[
(10) \quad \rho_{n+1} \leq (1 - \gamma_n)\rho_n + C_n^2d^2.
\]
The inequality (10) and a simple induction now yield
\[
(11) \quad 0 \leq \rho_n \leq B^2\alpha_n \quad \text{for all } n \geq 1,
\]
where $\alpha_n \geq 0$ is recursively generated by
\begin{equation}
\alpha_{n+1} = (1 - \gamma_n)\alpha_n + C_n^2, \quad \alpha_1 = 1,
\end{equation}
and $B^2 = \max\{\rho_1, d^2\}$.

Observe that $1 - \gamma_n = [1 - (1 - k)C_n]^2$ so that
\[ \gamma_n = (1 - k)C_n[2 - (1 - k)C_n] \]
and
\begin{equation}
\sum_{n=1}^{\infty} \gamma_n = 2(1 - k) \sum_{n=1}^{\infty} C_n - (1 - k)^2 \sum_{n=1}^{\infty} C_n^2 = \infty.
\end{equation}
Furthermore, $\sum_{n=1}^{\infty} C_n^2 < \infty$ implies $\lim_{n \to \infty} C_n = 0$.

Consequently, there is a sufficiently large $N$ such that $n \geq N$ implies $\gamma_n \in [0, 1]$. For $j \geq 1$, put
\[ \beta_j = \alpha_{N+j}, \delta_j = \gamma_{N+j}, \text{ and } \sigma_j = C_{N+j}. \]
Observe that (iii) implies
\begin{equation}
\sum_{j=1}^{\infty} \sigma_j^2 = \sum_{j=1}^{\infty} C_{N+j}^2 < \infty.
\end{equation}
So, from $\beta_1 = \alpha_{N+1} \geq 0$, (13), and (14), it follows from Lemma 2 that $\alpha_n \to 0$ as $n \to \infty$, so that (11) implies $\rho_n \to 0$ as $n \to \infty$, i.e., $\|x_n - p\| \to 0$ as $n \to \infty$, so that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p$.

REMARK 2. It is a consequence of the above proof that, under the hypotheses of the theorem, the fixed point of $T$ must be unique. The element $p \in F(T)$, where $F(T)$ denotes the set of fixed points of $T$, was arbitrarily chosen. Suppose now there is a $p^* \in F(T)$ with $p^* \neq p$. Repeating the argument of the theorem relative to $p^*$, one sees that (8) converges to both $p^*$ and $p$, showing that $F(T) = \{p\}$.

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REFERENCES


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