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THE MEAN CURVATURE OF A SET OF FINITE PERIMETER

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ABSTRACT. It is shown that an arbitrary set of finite perimeter in $\mathbf{R}^n$ minimizes some prescribed mean curvature functional given by an L^1 function on $\mathbf{R}^n$.

1. Introduction. A (Lebesgue) measurable set E of $\mathbf{R}^n$ is said to be of *finite perimeter* if and only if the distributional gradient $D\varphi_E$ of its characteristic function φ_E is a Radon vector measure on $\mathbf{R}^n$, with finite total variation $|D\varphi_E|$, i.e. iff

$$|D\varphi_E|(\mathbf{R}^n) \equiv \sup \left\{ \int_E \operatorname{div} G(x)\, dx\colon\ G \in C_0^1(\mathbf{R}^n, \mathbf{R}^n), |G(x)| \leq 1\ \forall x \right\} < +\infty.$$

Notice that $|D\varphi_E|(\mathbf{R}^n) = |D\varphi_{\mathbf{R}^n \setminus E}|(\mathbf{R}^n)$, and that $|D\varphi_E|(\mathbf{R}^n) = |D\varphi_F|(\mathbf{R}^n)$ whenever $|E\setminus F| + |F \setminus E| = 0$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbf{R}^n$. On the basis of the preceding remark, in the sequel we shall not distinguish between any two sets differing by a null set. In particular, the inclusion $E \subset F$ will mean $|E\setminus F| = 0$.

Sets of finite perimeter were introduced by E. De Giorgi in the fifties. He also proved the basic compactness and semicontinuity results, which allow the use of the direct methods in the treatment of some classical problems of the Calculus of Variations, such as the Plateau problem and the isoperimetric problem. We refer to [**1**, **2**, **5**] for a more detailed account. In addition, De Giorgi proved that the boundary of any set of *least* perimeter relative to another open set is an analytic manifold, except possibly for a closed singular set Σ. After the work of F. Almgren, J. Simons, H. Federer and E. Bombieri, E. De Giorgi, and E. Giusti, it is now well known that the Hausdorff dimension of Σ does not exceed $n - 8$. Later, minima of more general functionals were considered by U. Massari. For a given $H \in L^1(\mathbf{R}^n)$, Massari's functional reads as follows:

$$\mathcal{F}_H(E) = |D\varphi_E|(\mathbf{R}^n) + \int_E H(x)\, dx, \tag{1}$$

where E is an arbitrary set of finite perimeter in $\mathbf{R}^n$. The existence of minimizers of (1), with suitable "boundary conditions," is shown in [**3**, Theorem 1.1]. The rest of the paper [**3**] and the subsequent paper [**4**] deal with the regularity of the solutions, when H is bounded or in some L^p with $p > n$, respectively.

It can be easily seen (by computing the first variation of the functional (1)), that if E minimizes $\mathcal{F}_H$ with respect to compact perturbations, if H is continuous at $x \in \partial E$, and ∂E is smooth near x, then the value of the mean curvature of ∂E at x is given by $-H(x)/(n-1)$. This is the reason why the minimizers of (1) are

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usually called "sets of (generalized) mean curvature H." It is the aim of the present paper to show that, in the preceding sense, *every set of finite perimeter in* $\mathbf{R}^n$ *has mean curvature in* $L^1(\mathbf{R}^n)$, that is, we shall prove the following

THEOREM. *For every set E of finite perimeter in $\mathbf{R}^n$, there exists a function $H \in L^1(\mathbf{R}^n)$ such that $\mathcal{F}_H(E) \le \mathcal{F}_H(F)$ holds for every F of finite perimeter in $\mathbf{R}^n$.*

2. Proof of the Theorem. First of all, we notice that if, for the given set E, a function $H \in L^1(\mathbf{R}^n)$ can be found such that

$$\mathcal{F}_H(E) \le \mathcal{F}_H(F) \tag{2}$$

holds for every F with *either* $F \subset E$ *or* $E \subset F$, then the Theorem is proved, i.e. (2) holds *for every* $F \subset \mathbf{R}^n$ (we tacitly assume that all sets to be considered are of finite perimeter in $\mathbf{R}^n$). This can be easily seen, by adding the inequalities (2) corresponding to the test sets $E \cap F$ and $E \cup F$, and recalling that

$$|D\varphi_{E\cap F}|(\mathbf{R}^n) + |D\varphi_{E\cup F}|(\mathbf{R}^n) \le |D\varphi_E|(\mathbf{R}^n) + |D\varphi_F|(\mathbf{R}^n) \tag{3}$$

(see [**5**, 2.1.2(10)]). Thus, all we need is to define H on E in such a way that H is summable and (2) holds $\forall F \subset E$.

Step 1. We fix a measurable function $h(x)$ s.t. $h > 0$ on E and $\int_E h(x)\,dx < +\infty$, and denote by α the (positive and totally finite) measure

$$\alpha(F) = \int_F h(x)\,dx, \qquad F \subset E. \tag{4}$$

Clearly, $\alpha(F) = 0$ iff $|F| = 0$. For $\lambda > 0$ and $F \subset E$ we then consider the functional

$$\mathcal{F}_\lambda(F) = |D\varphi_F|(\mathbf{R}^n) + \lambda\alpha(E\backslash F). \tag{5}$$

By known results, every minimizing sequence is compact in $L^1_{\text{loc}}(\mathbf{R}^n)$; the functional being lower-semicontinuous with respect to the same convergence, we get, for every $\lambda > 0$, a solution E_λ to the problem: $\mathcal{F}_\lambda(F) \to \min$, $F \subset E$.

We choose a sequence $\{\lambda_i\}$ of positive numbers, strictly increasing to $+\infty$, and denote the corresponding solutions by $E_i \equiv E_{\lambda_i}$, so that $\forall i \ge 1$:

$$\mathcal{F}_{\lambda_i}(E_i) \le \mathcal{F}_{\lambda_i}(F) \quad \forall F \subset E. \tag{6}$$

If $i < j$, then, by adding the inequalities (6) corresponding to the index i and the test set $F = E_i \cap E_j$ and, respectively, to j and $F = E_i \cup E_j$, and recalling (3), we get immediately $(\lambda_j - \lambda_i)\alpha(E_i\backslash E_j) = 0$, hence $E_i \subset E_j$. Thus, the sequence of minimizers $\{E_i\}$ is increasing. On the other hand, using E as test set in (6) we get

$$|D\varphi_{E_i}|(\mathbf{R}^n) + \lambda_i\alpha(E\backslash E_i) \le |D\varphi_E|(\mathbf{R}^n) \quad \forall i \ge 1$$

from which we conclude that E_i converges monotonically and in $L^1_{\text{loc}}(\mathbf{R}^n)$ to E. Moreover, by semicontinuity we get

$$|D\varphi_E|(\mathbf{R}^n) = \lim_{i\to+\infty} |D\varphi_{E_i}|(\mathbf{R}^n). \tag{7}$$

Step 2. Now we put $\lambda_0 = 0$ and $E_0 = \varnothing$, and define

$$H(x) = \begin{cases} -\lambda_i \cdot h(x) & \text{if } x \in E_i\backslash E_{i-1}\ (i \ge 1), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Clearly, H is negative (almost everywhere) on E, and

$$\int_{\mathbf{R}^n} |H(x)|\,dx = \sum_{i=0}^{\infty} \lambda_{i+1}\alpha(E_{i+1}\backslash E_i). \tag{9}$$

Using E_{i+1} as test set in (6) we get

$$\lambda_i\alpha(E_{i+1}\backslash E_i) \le |D\varphi_{E_{i+1}}|(\mathbf{R}^n) - |D\varphi_{E_i}|(\mathbf{R}^n),$$

which holds $\forall i \ge 0$, whence

$$\sum_{i=0}^{\infty} \lambda_i\alpha(E_{i+1}\backslash E_i) \le |D\varphi_E|(\mathbf{R}^n) \tag{10}$$

by virtue of (7). At this point we make the additional assumption that $0 < \lambda_{i+1} - \lambda_i \le c\ \forall i \ge 0$, where c is a constant independent of i. Then clearly

$$\sum_{i=0}^{\infty}(\lambda_{i+1} - \lambda_i)\alpha(E_{i+1}\backslash E_i) \le c\alpha(E)$$

which, together with (9), (10), implies that

$$\int_{\mathbf{R}^n} |H(x)|\,dx \le |D\varphi_E|(\mathbf{R}^n) + c\alpha(E) < +\infty.$$

In conclusion, $H \in L^1(\mathbf{R}^n)$.

Step 3. We claim that for every $i \ge 1$ the inequality

$$|D\varphi_{E_i}|(\mathbf{R}^n) \le |D\varphi_F|(\mathbf{R}^n) + \sum_{j=1}^{i} \lambda_j\alpha((E_j\backslash E_{j-1})\backslash F) \tag{11}$$

holds $\forall F \subset E$.

We prove it by induction. When $i = 1$, (11) follows immediately from (6). Assume that (11) holds for a fixed $i \ge 1$ and every $F \subset E$. Using $F \cap E_i$ as test set, we get

$$|D\varphi_{E_i}|(\mathbf{R}^n) \le |D\varphi_{F\cap E_i}|(\mathbf{R}^n) + \sum_{j=1}^{i} \lambda_j\alpha((E_j\backslash E_{j-1})\backslash F) \tag{12}$$

while

$$\begin{aligned} &|D\varphi_{E_{i+1}}|(\mathbf{R}^n) + \lambda_{i+1}\alpha(E\backslash E_{i+1}) \\ &\quad\le |D\varphi_{F\cup E_i}|(\mathbf{R}^n) + \lambda_{i+1}\alpha((E\backslash E_{i+1})\backslash F) + \lambda_{i+1}\alpha((E_{i+1}\backslash E_i)\backslash F) \end{aligned} \tag{13}$$

since E_{i+1} is a minimizer of $\mathcal{F}_{\lambda_{i+1}}$. By adding (12) and (13), and simplifying (recall again (3)), we obtain (11) with i replaced by $i + 1$. Our assertion is therefore confirmed.

Letting i tend to infinity (for a fixed $F \subset E$) in (11) we obtain, on account of (7), (4), and (8):

$$|D\varphi_E|(\mathbf{R}^n) \le |D\varphi_F|(\mathbf{R}^n) + \sum_{j=1}^{\infty} \lambda_j\alpha((E_j\backslash E_{j-1})\backslash F) = |D\varphi_F|(\mathbf{R}^n) - \int_{E\backslash F} H(x)\,dx.$$

Therefore, (2) holds $\forall F \subset E$, and the proof of the Theorem is concluded.

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