THE MEAN CURVATURE
OF A SET OF FINITE PERIMETER
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ABSTRACT. It is shown that an arbitrary set of finite perimeter in $\mathbb{R}^n$ mini-
mizes some prescribed mean curvature functional given by an $L^1$ function on
$\mathbb{R}^n$.

1. Introduction. A (Lebesgue) measurable set $E$ of $\mathbb{R}^n$ is said to be of finite
perimeter if and only if the distributional gradient $D\varphi_E$ of its characteristic function
$\varphi_E$ is a Radon vector measure on $\mathbb{R}^n$, with finite total variation $|D\varphi_E|$, i.e. iff

$$|D\varphi_E|(\mathbb{R}^n) = \sup \left\{ \int_E \text{div} G(x) \, dx : G \in C_0^1(\mathbb{R}^n, \mathbb{R}^n), |G(x)| \leq 1 \forall x \right\} < +\infty.$$ 

Notice that $|D\varphi_E|(\mathbb{R}^n) = |\mathcal{D}\chi_E|$, and that $|D\varphi_E|(\mathbb{R}^n) = |D\varphi_F|(\mathbb{R}^n)$
whenever $|E \setminus F| + |F \setminus E| = 0$, where $| \cdot |$ denotes the Lebesgue measure on $\mathbb{R}^n$. On
the basis of the preceding remark, in the sequel we shall not distinguish between
any two sets differing by a null set. In particular, the inclusion $E \subseteq F$ will mean
$|E \setminus F| = 0$.

Sets of finite perimeter were introduced by E. De Giorgi in the fifties. He also
proved the basic compactness and semicontinuity results, which allow the use of
the direct methods in the treatment of some classical problems of the Calculus of
Variations, such as the Plateau problem and the isoperimetric problem. We refer
to [1, 2, 5] for a more detailed account. In addition, De Giorgi proved that the
boundary of any set of least perimeter relative to another open set is an analytic
manifold, except possibly for a closed singular set $\Sigma$. After the work of F. Almgren,
J. Simons, H. Federer and E. Bombieri, E. De Giorgi, and E. Giusti, it is now well
known that the Hausdorff dimension of $\Sigma$ does not exceed $n - 8$. Later, minima of
more general functionals were considered by U. Massari. For a given $H \in L^1(\mathbb{R}^n)$,
Massari's functional reads as follows:

$$\mathcal{F}_H(E) = |D\varphi_E|(\mathbb{R}^n) + \int_E H(x) \, dx,$$

where $E$ is an arbitrary set of finite perimeter in $\mathbb{R}^n$. The existence of minimizers
of (1), with suitable "boundary conditions," is shown in [3, Theorem 1.1]. The
rest of the paper [3] and the subsequent paper [4] deal with the regularity of the
solutions, when $H$ is bounded or in some $L^p$ with $p > n$, respectively.

It can be easily seen (by computing the first variation of the functional (1)),
that if $E$ minimizes $\mathcal{F}_H$ with respect to compact perturbations, if $H$ is continuous
at $x \in \partial E$, and $\partial E$ is smooth near $x$, then the value of the mean curvature of $\partial E$
at $x$ is given by $-H(x)/(n - 1)$. This is the reason why the minimizers of (1) are
usually called “sets of (generalized) mean curvature $H$.” It is the aim of the present paper to show that, in the preceding sense, every set of finite perimeter in $\mathbb{R}^n$ has mean curvature in $L^1(\mathbb{R}^n)$, that is, we shall prove the following

**Theorem.** For every set $E$ of finite perimeter in $\mathbb{R}^n$, there exists a function $H \in L^1(\mathbb{R}^n)$ such that $\mathcal{F}_H(E) \leq \mathcal{F}_H(F)$ holds for every $F$ of finite perimeter in $\mathbb{R}^n$.

2. **Proof of the Theorem.** First of all, we notice that if, for the given set $E$, a function $H \in L^1(\mathbb{R}^n)$ can be found such that

(2) $\mathcal{F}_H(E) \leq \mathcal{F}_H(F)$

holds for every $F$ with either $F \subset E$ or $E \subset F$, then the Theorem is proved, i.e. (2) holds for every $F \subset \mathbb{R}^n$ (we tacitly assume that all sets to be considered are of finite perimeter in $\mathbb{R}^n$). This can be easily seen, by adding the inequalities (2) corresponding to the test sets $E \cap F$ and $E \cup F$, and recalling that

(3) $|D\phi_{E \cap F}|(\mathbb{R}^n) + |D\phi_{E \cup F}|(\mathbb{R}^n) \leq |D\phi_E|(\mathbb{R}^n) + |D\phi_F|(\mathbb{R}^n)$

(see [5, 2.1.2(10)]). Thus, all we need is to define $H$ on $E$ in such a way that $H$ is summable and (2) holds $\forall F \subset E$.

**Step 1.** We fix a measurable function $h(x)$ s.t. $h > 0$ on $E$ and $\int_E h(x) \, dx < +\infty$, and denote by $\alpha$ the (positive and totally finite) measure

(4) $\alpha(F) = \int_F h(x) \, dx, \quad F \subset E$.

Clearly, $\alpha(F) = 0$ iff $|F| = 0$. For $\lambda > 0$ and $F \subset E$ we then consider the functional

(5) $\mathcal{F}_\lambda(F) = |D\phi_F|(\mathbb{R}^n) + \lambda \alpha(E \setminus F)$.

By known results, every minimizing sequence is compact in $L^1_{\text{loc}}(\mathbb{R}^n)$; the functional being lower-semicontinuous with respect to the same convergence, we get, for every $\lambda > 0$, a solution $E_\lambda$ to the problem: $\mathcal{F}_\lambda(F) \to \min$, $F \subset E$.

We choose a sequence $\{\lambda_i\}$ of positive numbers, strictly increasing to $+\infty$, and denote the corresponding solutions by $E_i \equiv E_{\lambda_i}$, so that $\forall i \geq 1$:

(6) $\mathcal{F}_{\lambda_i}(E_i) \leq \mathcal{F}_{\lambda_i}(F) \quad \forall F \subset E$.

If $i < j$, then, by adding the inequalities (6) corresponding to the index $i$ and the test set $F = E_i \cap E_j$ and, respectively, to $j$ and $F = E_i \cup E_j$, and recalling (3), we get immediately $(\lambda_j - \lambda_0) \alpha(E_i \setminus E_j) = 0$, hence $E_i \subset E_j$. Thus, the sequence of minimizers $\{E_i\}$ is increasing. On the other hand, using $E$ as test set in (6) we get

$|D\phi_{E_i}|(\mathbb{R}^n) + \lambda_i \alpha(E \setminus E_i) \leq |D\phi_E|(\mathbb{R}^n) \quad \forall i \geq 1$

from which we conclude that $E_i$ converges monotonically and in $L^1_{\text{loc}}(\mathbb{R}^n)$ to $E$. Moreover, by semicontinuity we get

(7) $|D\phi_E|(\mathbb{R}^n) = \lim_{i \to +\infty} |D\phi_{E_i}|(\mathbb{R}^n)$.

**Step 2.** Now we put $\lambda_0 = 0$ and $E_0 = \emptyset$, and define

(8) $H(x) = \begin{cases} -\lambda_i \cdot h(x) & \text{if } x \in E_i \setminus E_{i-1} \quad (i \geq 1), \\ 0 & \text{otherwise.} \end{cases}$
Clearly, $H$ is negative (almost everywhere) on $E$, and

\[ \int_{\mathbb{R}^n} |H(x)| \, dx = \sum_{i=0}^{\infty} \lambda_{i+1} \alpha(E_{i+1} \setminus E_i). \]

Using $E_{i+1}$ as test set in (6) we get

\[ \lambda_i \alpha(E_{i+1} \setminus E_i) \leq |D\varphi_{E_{i+1}}|(\mathbb{R}^n) - |D\varphi_{E_{i}}|(\mathbb{R}^n), \]

which holds $\forall i \geq 0$, whence

\[ \sum_{i=0}^{\infty} \lambda_i \alpha(E_{i+1} \setminus E_i) \leq |D\varphi_E|(\mathbb{R}^n) \]

by virtue of (7). At this point we make the additional assumption that $0 < \lambda_{i+1} - \lambda_i \leq c \forall i \geq 0$, where $c$ is a constant independent of $i$. Then clearly

\[ \sum_{i=0}^{\infty} (\lambda_{i+1} - \lambda_i) \alpha(E_{i+1} \setminus E_i) \leq c\alpha(E) \]

which, together with (9), (10), implies that

\[ \int_{\mathbb{R}^n} |H(x)| \, dx \leq |D\varphi_E|(\mathbb{R}^n) + c\alpha(E) < +\infty. \]

In conclusion, $H \in L^1(\mathbb{R}^n)$.

\textit{Step 3.} We claim that for every $i \geq 1$ the inequality

\[ |D\varphi_{E_i}|(\mathbb{R}^n) \leq |D\varphi_F|(\mathbb{R}^n) + \sum_{j=1}^{i} \lambda_j \alpha((E_j \setminus E_{j-1}) \setminus F) \]

holds $\forall F \subset E$.

We prove it by induction. When $i = 1$, (11) follows immediately from (6). Assume that (11) holds for a fixed $i \geq 1$ and every $F \subset E$. Using $F \cap E_i$ as test set, we get

\[ |D\varphi_{E_i}|(\mathbb{R}^n) \leq |D\varphi_{F \cap E_i}|(\mathbb{R}^n) + \sum_{j=1}^{i} \lambda_j \alpha((E_j \setminus E_{j-1}) \setminus F) \]

while

\[ |D\varphi_{E_{i+1}}|(\mathbb{R}^n) + \lambda_{i+1} \alpha(E \setminus E_{i+1}) \leq |D\varphi_{F \cup E_i}|(\mathbb{R}^n) + \lambda_{i+1} \alpha((E \setminus E_{i+1}) \setminus F) + \lambda_{i+1} \alpha((E_{i+1} \setminus E_i) \setminus F) \]

since $E_{i+1}$ is a minimizer of $\mathcal{J}_{i+1}$. By adding (12) and (13), and simplifying (recall again (3)), we obtain (11) with $i$ replaced by $i + 1$. Our assertion is therefore confirmed.

Letting $i$ tend to infinity (for a fixed $F \subset E$) in (11) we obtain, on account of (7), (4), and (8):

\[ |D\varphi_E|(\mathbb{R}^n) \leq |D\varphi_F|(\mathbb{R}^n) + \sum_{j=1}^{\infty} \lambda_j \alpha((E_j \setminus E_{j-1}) \setminus F) = |D\varphi_F|(\mathbb{R}^n) - \int_{E \setminus F} H(x) \, dx. \]

Therefore, (2) holds $\forall F \subset E$, and the proof of the Theorem is concluded.
REFERENCES


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