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SPECTRA OF THE DIFFERENCE AND PRODUCT OF PROJECTIONS

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ABSTRACT. A short natural proof of the known relation between spectra of the difference and the product of two orthogonal projections is given. This proof also generalizes this result.

The problem of how to determine the spectrum of the product of two orthogonal projections in an arbitrary complex Hilbert space H from the spectrum of their difference arises from statistics and was recently solved for point spectrum by W. N. Anderson, Jr., E. J. Harner, and G. E. Trapp [**1**]. Here is their result, stated for the whole spectrum.

THEOREM. *For any pair of orthogonal projections P and Q on H the spectrum of the product PQ lies in the interval* $[0,1]$ *and*

$$\sigma(Q-P)\backslash\{-1,0,1\}=\{\pm(1-\mu)^{1/2};\mu\in\sigma(PQ)\backslash\{0,1\}\}.$$

REMARK. Note that this Theorem can be equivalently formulated as follows: For any pair of orthogonal projections P and Q on H the spectrum of $Q-P$ lies in the interval $[-1,1]$, $\sigma(Q-P)\backslash\{-1,0,1\}=\sigma(P-Q)\backslash\{-1,0,1\}$, and

$$\sigma(PQ)\backslash\{0,1\}=\{1-\lambda^2;\lambda\in\sigma(Q-P)\backslash\{-1,0,1\}\}.$$

PROOF. Write $U=\operatorname{Im}P$, $V=\operatorname{Ker}P$, decompose the space H into $H=U\oplus V$, and represent the operator Q as an operator matrix (see [**2**] for definition of operator matrices)

$$Q=\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

Note that A is a selfadjoint operator on U, C is a selfadjoint operator on V, and that $\sigma(A)\subset[0,1]$ and $\sigma(C)\subset[0,1]$. From $Q^2=I$ we get

$$BB^*=A-A^2,\quad B^*B=C-C^2,\quad \text{and}\quad AB+BC=B$$

which implies that $\operatorname{Ker}B\subset V$ is invariant under C and must therefore reduce this operator. Moreover, it is clear for the restriction $C|_{\operatorname{Ker}B}$ that $\sigma(C|_{\operatorname{Ker}B})\subset\{0,1\}$. Similarly, the subspace $\operatorname{Ker}B^*\subset U$ reduces A and $\sigma(A|_{\operatorname{Ker}B^*})\subset\{0,1\}$. Denote

$$A_1=A|_{\overline{\operatorname{Im}B}},\quad C_1=C|_{\overline{\operatorname{Im}B^*}},\quad \text{and}\quad B_1=B|_{\overline{\operatorname{Im}B^*}};$$

then, $B_1\colon\overline{\operatorname{Im}B^*}\to\overline{\operatorname{Im}B}$ is a quasiaffinity, i.e. a bounded injective operator with dense range. In addition, B_1 intertwines C_1 and $I-A_1$, i.e. $B_1C_1=(I-A_1)B_1$. It is well known that this forces the spectral measures E of C_1 and F of $I-A_1$ to satisfy

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the relation $B_1 E(\sigma) = F(\sigma) B_1$ for any interval σ of the real line. Consequently, the projections $E(\sigma)$ and $F(\sigma)$ are zero together which yields $\sigma(C_1) = \sigma(I - A_1)$. It follows that $\sigma(C)\backslash\{0,1\} = \sigma(I - A)\backslash\{0,1\}$.

The form which the matrix $(\lambda - Q + P)^{-1}$ takes in the 2-dimensional case, where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \cos^2 \vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & \sin^2\vartheta \end{bmatrix},$$

suggests our defining the following operator matrix $R(\lambda)$ for every complex λ for which it makes sense:

$$R(\lambda) = \begin{bmatrix} I + (\lambda - \lambda^2)(A - I + \lambda^2)^{-1} & B(\lambda^2 - C)^{-1} \\ B^*(A - I + \lambda^2)^{-1} & -I + (\lambda + \lambda^2)(\lambda^2 - C)^{-1} \end{bmatrix}.$$

Use the above relations between the operators A, B, and C to verify

$$(\lambda - Q + P)R(\lambda) = R(\lambda)(\lambda - Q + P) = I.$$

Representing also the operator PQ by the operator matrix

$$PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

we see that $\sigma(PQ)\backslash\{0\} = \sigma(A)\backslash\{0\}$. Take any $\mu \in \rho(PQ)$, $\mu \notin \{0,1\}$, to see that $\mu \in \rho(A)$ and $1 - \mu \in \rho(C)$. Put either $\lambda = (1-\mu)^{1/2}$, or $\lambda = -(1-\mu)^{1/2}$ into $R(\lambda)$ to get that both belong to $\rho(Q - P)$. To obtain the inverse inclusion, take any $\lambda \in \rho(Q - P)$, $\lambda \notin \{-1, 0, 1\}$, and note that $R(\mu)$ converges to $(\lambda - P + Q)^{-1}$ when μ approaches the point λ through complex values. It follows that $(A - I + \lambda^2)^{-1}$ exists and that

$$(A - I + \lambda^2)^{-1} = (R(\lambda)^P - I)/(\lambda - \lambda^2),$$

where $R(\lambda)^P$ denotes the compression of $R(\lambda)$ to the space U. Therefore, $1 - \lambda^2 \in \rho(A)$ and consequently $1 - \lambda^2 \in \rho(PQ)$.

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