A MAXIMUM PRINCIPLE
FOR QUOTIENT NORMS IN $H^\infty$

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ABSTRACT. Let $G$ be a closed subset of the open unit disk $D$ in the complex plane, and let $p$ denote a general polynomial of degree $n$ which has all of its roots in $G$. For a fixed $h$ in $H^\infty$, $\|h - pH^\infty\|_{H^\infty/pH^\infty}$ is maximized only if all the zeros of $p$ are on the boundary of $G$.

In studying a problem on the spectral radius of matrices, V. Pták was led to the following extremal problem: Let $H^\infty$ denote the space of bounded analytic functions on the open unit disk $D$ of the complex plane $C$, and let $h$ be a fixed function in $H^\infty$. Among all polynomials $p$ of degree $n$ whose zeros are in $\{z : |z| \leq r\}$ for a fixed $r < 1$, find one which maximizes $\|h - pH^\infty\|$ in the quotient space $H^\infty/pH^\infty$ (see [2, 5]). Actually, Pták considered the case when $h$ is of the form $h(z) = z^m$ for a fixed integer $m$. In fact, he showed that in the case $m = n$, the extremal polynomial can be taken to be $(z - r)^n$. It was conjectured that this is true for $m > n$ as well as that, for the general $h$ in $H^\infty$, each extremal $p$ has all of its zeros on the circle $\{|z| = r\}$. The latter conjecture was recently proved by N. J. Young [4] in the special case that $h$ is a Blaschke product of degree $n$, though the conjecture remained open even in the case $h = z^m$ for $m > n$. The contribution of this paper is to prove the following maximum principle for the extremal polynomial.

**Theorem 1.** Let $G$ be a closed subset of $D$ and let $p$ denote a general polynomial of degree $n$ which has all of its roots in $G$. For a fixed $h$ in $H^\infty$, let $F(p) = \|h - pH^\infty\|_{H^\infty/pH^\infty}$. If $F$ is not constant as $p$ varies, then it attains its maximum at $p$ only if all the zeros of $p$ lie on the boundary of $G$.

The work of Pták and Young mentioned above has been largely operator-theoretic. In contrast, the present treatment is completely elementary, relying on the Schur algorithm for the solution of the Nevanlinna-Pick interpolation problem (see [1, 3]). Since the treatment here is somewhat nonstandard, a brief description of the Schur algorithm is given below.

Let $a_1, a_2, \ldots, a_n$ be points in $D$ and $W_1, W_2, \ldots, W_n$ the values to be interpolated along the $a_j$ by a function $f$ in the unit ball $\Sigma$ of $H^\infty$. We allow repetitions in the $a_j$'s as long as they occur consecutively. For each $k$, let $d_k$ denote the number of times $a_j = a_k$ for $j < k$. We are looking for a function $f$ in $\Sigma$ which satisfies

$$f^{(d_k)}(a_k) = w_k, \quad k = 1, 2, \ldots, n.$$

The Schur algorithm proceeds inductively as follows. Suppose that $f$ is in $\Sigma$ and fits the given data. Take $a_1$. If $|w_1| > 1$, no solution exists. If $|w_1| \leq 1$, the function

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$f_1$ defined by

\begin{equation}
(1) \quad f_1 = \frac{f - w_1}{z - a_1} \cdot \frac{1 - \bar{a}_1 z}{1 - \bar{w}_1 f}
\end{equation}

belongs to $\Sigma$. Writing $b_1 = (z - a_1)/(1 - \bar{a}_1 z)$, we get that

\begin{equation}
(2) \quad f = \frac{b_1 f_1 + w}{1 + \bar{w}_1 b_1 f_1}
\end{equation}

and no matter what $f_1$ in $\Sigma$ is, the right hand expression above reduces to $w_1$ when evaluated at $z = a_1$. It is easily checked that, for any positive integer $k$, the first $k$ derivatives of $f$ at $a_1$ can be determined by the first $k - 1$ derivatives of $f_1$ at $a_1$ and vice-versa. Likewise, at any other node, the interpolation data for $f$ determine interpolation data for $f_1$ and vice-versa. So the problem of finding $f$ in $\Sigma$ reduces to solving a lower order interpolation for $f_1$ with revised data. If one proceeds inductively, there are three possibilities.

(i) The process reveals at some point that no solution exists in $\Sigma$.

(ii) The process terminates at the $j$th stage ($0 < j < n - 1$) yielding a unique solution. This solution is a Blaschke product of degree $j$. Conversely, if a Blaschke product of order $j \leq n - 1$ is among the solutions, then the process terminates at the $j$th stage.

(iii) The process can be carried through the $n$th stage in which case the choice of $f_n$ is indeterminate.

Now suppose the interpolation problem has a solution in $\Sigma$ for the data $\{a_1, \ldots, a_{n-1}; w_1, \ldots, w_{n-1}\}$. Let $a_n$ in $D$ be added. Then the set of possible $w_n$ for which the problem with data $\{a_1, \ldots, a_n; w_1, \ldots, w_n\}$ can be solved in $\Sigma$ is a closed disk whose center and radius are determined by $a_1, a_2, \ldots, a_n$ and $w_1, w_2, \ldots, w_{n-1}$. The augmented interpolation problem has a unique solution if and only if $w_n$ belongs to the boundary of that disk (see [3] or Chapter 1 of [1]). The disk reduces to a point if and only if condition (ii) is met on or before the $(n - 2)$th stage.

We need to take a closer look at the Schur algorithm. Let

$$
\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n) = \{f \in \Sigma: f^{(d_k)}(a_k) = w_k, k = 1, 2, \ldots, n\}
$$

and

$$
D(a_1, \ldots, a_{k+1}; w_1, \ldots, w_k) = \{f^{(d_{k+1})}(a_{k+1}): f \in \Sigma(a_1, \ldots, a_k; w_1, \ldots, w_k)\}.
$$

Then, from (1) and (2),

$$
\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n) = \left\{ f = \frac{b_1 \varphi + w_1}{1 + \bar{w}_1 b_1 \varphi}: \varphi \in \Sigma(a_2, \ldots, a_n; \hat{w}_2, \ldots, \hat{w}_n) \right\},
$$

where the $\hat{w}_j$ are computed as follows:

**Case 1.** ($a_j \neq a_1$). The first $d$ derivatives of $\varphi$ at $a_j$ are determined by the first $d$ derivatives of $f$ and vice-versa. Thus,

$$
\hat{w}_j = [(f - w_1)/(1 - \bar{w}_1 f)]^{(d_j)}(a_j)
$$

for any $f$ in $\Sigma(a_1, \ldots, a_j; w_1, \ldots, w_j)$.

**Case 2.** ($a_j = a_1$). From (1), letting $g = (1 - \bar{a}_1 z)/(1 - \bar{w}_1 f)$, we have

$$
\hat{w}_j = \sum_{m=0}^{j-1} \binom{j-1}{m} f^{(m+1)}(a_1) g^{(j-1-m)}(a_1) = \sum_{m=0}^{j-1} \binom{j-1}{m} w_{m+1} g^{(j-1-m)}(a_1).
$$
Hence, in either case, if $\Sigma(a_1, \ldots, a_j; w_1, \ldots, w_j)$ is nontrivial (i.e., contains more than one function), then $\hat{w}_2, \ldots, \hat{w}_j$ vary continuously with $w_1, \ldots, w_j$.

Though it is not used in the proof of Theorem 1, the following proposition is of interest in its own right.

**PROPOSITION 1.** Suppose $D(a_1, \ldots, a_{k+1}; w_1, \ldots, w_k)$ has nonempty interior. Then it is a disk whose center and radius vary continuously with $w_1, w_2, \ldots, w_k$.

**PROOF.** (INDUCTION ON $k$). For $k = 1$, there are two cases:

If $a_2 \neq a_1$,

$$D(a_1, a_2; w_1) = \left\{ \frac{[b_1(a_2)w + w_1]}{[1 + \bar{w}_1 b_1(a_2)w]} : |w| \leq 1 \right\}.$$  

If $a_2 = a_1$,

$$D(a_1, a_1; w_1) = \left\{ \frac{w[1 - |w|^2]}{[1 - |a_1|^2]} : |w| < 1 \right\}.$$  

Now suppose that the lemma holds for $D_{j+1} = D(a_1, \ldots, a_{j+1}; w_1, \ldots, w_j)$ whenever $j < k$, and suppose that $D_{k+1}$ has nonempty interior. Then

$$D_{k+1} = \left\{ \frac{(d_{k+1})'(a_{k+1})}{f(a_1, \ldots, a_k; w_1, \ldots, w_k)} : f \in \Sigma(a_1, \ldots, a_k; w_1, \ldots, w_k) \right\}$$

$$= \left\{ \frac{[b_1\varphi + w_1]}{[1 + \bar{w}_1 b_1\varphi]} \right\}^{(d_{k+1})'(a_{k+1})}.$$

If $d_{k+1} = 0$, then $D_{k+1}$ is a Mobius transformation of $D(a_2, \ldots, a_{k+1}; w_2, \ldots, w_k)$ with parameter $w_1$ so that the desired result holds by the continuity of the $\hat{w}_j$ and the inductive hypothesis. For the case $1 \leq d_{k+1} < k$, let $f$ be a solution to the $k$th order interpolation problem, and let $\varphi$ be related to $f$ as in (3). Then

$$[1 + \bar{w}_1 b_1\varphi]f = b_1\varphi + w_1.$$ Suppressing the subscript $k+1$, we have

$$f^{(d)}(a) \left[1 + \bar{w}_1 b_1(a)\varphi(a)\right] = - \sum_{m=0}^{d-1} \left( \frac{d}{m} \right) f^{(m)}(a) \left[1 + \bar{w}_1 b_1\varphi \right]^{(d-m)}(a)$$

$$+ \sum_{m=0}^{d} \left( \frac{d}{m} \right) b_1^{(d-m)}(a)\varphi^{(m)}(a),$$

$$f^{(d_{k+1})}(a_{k+1}) = R + C\hat{w},$$

where $\hat{w} \in D(a_2, \ldots, a_{k+1}; \hat{w}_2, \ldots, \hat{w}_k)$ and where $R$ and $C$ are rational functions of $w_1, \ldots, w_k, \hat{w}_2, \ldots, \hat{w}_k$. By the inductive assumption and the continuity of the $\hat{w}_j$, the desired result again follows. The last remaining case $d_{k+1} = k$ is treated in a similar fashion.

The next two propositions are used in the proof of Theorem 1.

**PROPOSITION 2.** Suppose that $\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n)$ contains a Blaschke product $B$ of order $m \leq n - 1$. Then there exists $\delta_0 > 0$ such that whenever $|w_j - w'_j| < \delta < \delta_0$ for $j = 1, 2, \ldots, m$, then $\Sigma(a_1, \ldots, a_n; w_1', \ldots, w_n')$ contains a Blaschke product $b$ (which depends on the $w'_j$) of order $m$. Moreover, $b$ can be chosen so that $\|B - b\|_\infty \to 0$ as $\sup\{|w_j - w'_j| : 1 \leq j \leq m\} \to 0$.

**PROOF.** If $m = 1$, then

$$B = (b_1\hat{w}_2 + w_1)/(1 + \bar{w}_1 b_1\hat{w}_2).$$
so the desired conclusion follows from the continuity of $\tilde{w}_2$. We now proceed inductively. If $\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n)$ contains a Blaschke product of order $m \leq n - 1$, then $D(a_1, \ldots, a_{m+1}; w_1, \ldots, w_m)$ is a nondegenerate disk and $w_{m+1}$ belongs to its boundary. Also, $B = (b_1 B + w_1)/(1 + \tilde{w}_1 b_1 B)$, where $B$ is a Blaschke product of order $m - 1$ in $\Sigma(a_2, \ldots, a_{m+1}; \tilde{w}_2, \ldots, \tilde{w}_{m+1})$. By induction, there exists $\eta_0 > 0$ such that whenever $|\tilde{w}_j - \tilde{w}_j| < \eta < \eta_0$ for $j = 2, \ldots, m$, then there exists a Blaschke product $\hat{b}$ of order $m - 1$ in $\Sigma(a_2, \ldots, a_m; \tilde{w}'_2, \ldots, \tilde{w}'_m)$ and such that $\|B - \hat{b}\|_\infty \to 0$ as $sup\{|\tilde{w}_j - \tilde{w}'_j| : j = 2, \ldots, m\} \to 0$. The desired result now follows from the continuity of the $w_j$ as functions of $w_1, \ldots, w_j$.

Also needed will be the following well-known connection between interpolation and approximation theory. Suppose that $b$ is a Blaschke product with zero sequence $a_1, \ldots, a_n$ (all repetitions are assumed to be consecutive), and let $h$ be a function in $H^\infty$. Then a function $f$ in $H^\infty$ is said to interpolate $h$ along the zeros of $b$ if $f - h$ belongs to $bH^\infty$. Of course, this means that $h$ is a solution to the interpolation problem with data $\{a_1, \ldots, a_n; w_1, \ldots, w_n\}$ where each $w_k$ is a derivative of appropriate order of $h$ at $a_k$. Now let $d = dist(h, bH^\infty)$ which is defined by

$$dist(h, bH^\infty) = \inf\{\|h - bg\|_\infty : g \text{ is in } H^\infty\}.$$ 

Then $d$ is characterized in terms of interpolation of $h$ along the zeros of $b$ as follows.

**Proposition 3.** Let $h$ be in $H^\infty$ and let $b$ be a Blaschke product of order $n$. A positive number $c$ equals $dist(h, bH^\infty)$ if and only if $h/c$ can be interpolated along the zeros of $b$ by a Blaschke product $B$ of order at most $n - 1$. Alternatively, $dist(h, bH^\infty)$ can be characterized as the least real number $t \geq 0$ such that $h - tg$ belongs to $bH^\infty$ for some $g$ in $H$.

**Proof.** If $dist(h, bH^\infty) = 0$ there is nothing to prove. Suppose that $B$ is a Blaschke product of order $\leq n - 1$, and that $h/c - B$ is in $bH^\infty$. Assume that $d = dist(h, bH^\infty) < c$. Then there is a function $g$ in $H^\infty$ such that $\|h - bg\|_\infty < |cB|$ on the boundary of $D$. Thus, $cB - (h - bg)$ is in $bH^\infty$ and, by Rouché's theorem, has at most $n - 1$ zeros in $D$ counting multiplicity. This is absurd, so $c \leq d$. To establish the opposite inequality, note that $h - cB = bf$ for some $f$ in $H^\infty$. Thus, $c = \|h - bf\|_\infty \geq dist(h, bH^\infty) = d$. The proof is concluded by the observation that for $c = \inf\{r > 0 : (h/r - bH^\infty) \cap \Sigma \neq \emptyset\}$, there exists a Blaschke product $B$ of order $\leq n - 1$ such that $h/c - B$ is in $bH^\infty$ (this follows from the Schur algorithm).

Theorem 1 will now be proved by induction on the number of zeros of $p$. If $p(z) = z - a$, where $a$ is in the interior of $G$, then $dist(h, pH^\infty) = |h(a)|$ since $h - h(a) \in pH^\infty$ and, for any $g$ in $H^\infty$, $\|h - pg\|_\infty \geq |h(a)|$. If $h$ is not constant, then there is an $a'$ near $a$ in $G$ such $|h(a')| > |h(a)|$.

Assume now that the theorem has been established for all $p$ with at most $n - 1$ zeros. Let $p$ have zero sequence $a_1, \ldots, a_n$ in $G$ listed according to our convention on repetitions, and assume that $a_n$ is in the interior of $G$. We shall show that either $h = cB$, where $c$ is a constant and where $B$ is a Blaschke product of degree at most $n - 1$ (in which case $F(p) = c$ for all $p$ with $n$ zeros in $D$), or that $F(q) > F(p)$ for some polynomial $q$ of degree $n$ with zeros in $G$ near the zeros of $p$. Let $b$ be the Blaschke product with zero sequence $a_1, \ldots, a_n$ and let $d = F(p) = dist(h, bH^\infty)$. We may assume that $h$ is not identically zero. If $d = 0$, just perturb $a_n$ by a small amount to move it away from the zero set of $h$ while still remaining in $G$. For
d > 0, there exists a unique Blaschke product $B$ with at most $m \leq n - 1$ zeros which interpolates the function $h/d$ along the $a_j$. If $m < n - 1$, then we also have, by Proposition 3, that

$$d = \text{dist} \left( h, \left[ \prod_{k=n-m}^n (z - a_k) \right] H^\infty \right).$$

By the inductive assumption there exists a polynomial $p_1$ whose zeros lie in $G$ and such that $d$ is less than $\text{dist}(h, p_1 H^\infty)$. Now let $q$ be a polynomial obtained from $p_1$ by adjoining $n - m - 1$ zeros in $G$. Then $F(q) \geq F(p_1) > F(p)$. If $m = n - 1$, we have for each small $s > 0$, a Blaschke product $B_s$ of order $n - 1$ which interpolates $h/(d + s)$ along $a_1, \ldots, a_{n-1}$ and such that $\|B - B_s\|_\infty \to 0$ as $s \to 0$. Let $f_s = B_s - h/(d + s)$. Then $f_s \to B - h/d$ as $s \to 0$. If $a_k \neq a_{k+1} = \cdots = a_n$, then either $h = dB$ or, by Hurwitz' Theorem, some $f_s$ has $n - k$ zeros (counting multiplicities) in $G$ near $a_n$. Denote them by $a_{k+1}', \ldots, a_n'$. Letting $q$ be a polynomial of degree $n$ with zero set $\{a_1, \ldots, a_k, a_{k+1}', \ldots, a_n'\}$, we have, by Proposition 3, that $F(q) = d + s > d$.

Finally, to relate the work in this paper to the operator-theoretic context of Pták and Young's original conjecture, we have (see [2]), as a corollary to Theorem 1,

**Theorem 2.** If $h \in H^\infty$ and $0 < r < 1$, then among all $n \times n$ contractions $A$ with all eigenvalues in the disk $\{z : |z| \leq r\}$, $\|h(A)\|$ attains its maximum at a matrix $A$ having all of its eigenvalues on the circle $\{z : |z| = r\}$.

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**References**


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