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EXTREMAL MULTILINEAR FORMS ON BANACH SPACES

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ABSTRACT. Suppose that L is a continuous symmetric m-linear form defined on a complex Banach space E, and $\hat{L}$ is the associated homogeneous polynomial. If

$$\|L\| = (m^m/m!)\|\hat{L}\|,$$

we prove that E contains an almost isometric copy of l_m^1. In particular if E is an m-dimensional space, then E is isometrically isomorphic to l_m^1. We also prove that the only examples of such extremal L which achieve their norm are suitable "extensions " of a known example given by Nachbin.

Throughout this paper K denotes either the complex field **C** or the real field **R**. Given any index set Γ we denote by $l^1(\Gamma)$ the collection of all K-valued families $x = (x_i)$ such that

$$\|x\| := \sum_{\Gamma} |x_i|$$

is finite. If Γ is the set of positive integers we denote $l^1(\Gamma)$ by l^1, while if $\Gamma = \{1, \dots, n\}$, where n is a positive integer, we denote $l^1(\Gamma)$ by l_n^1. If E is a vector space over the field K we write E^m for the product $E \times \cdots \times E$ with m-factors. An m-linear form $L\colon E^m \to K$ is said to be symmetric if

$$L(x_1, \dots, x_m) = L(x_{\sigma(1)}, \dots, x_{\sigma(m)})$$

for any $x_1, \dots, x_m$ in E and any permutation σ of the first m natural numbers.

If E is a normed space over K, we denote by $\mathscr{L}_m^s(E, K)$ the space of all continuous symmetric m-linear forms $L\colon E^m \to K$. A mapping $P\colon E \to K$ is said to be a homogeneous polynomial of degree m if $P = \hat{L}$ for some $L \in \mathscr{L}_m^s(E, K)$, where $\hat{L}$ is defined by

$$\hat{L}(x) = L(x, \dots, x).$$

If $L \in \mathscr{L}_m^s(E, K)$ we define the norms of $\hat{L}$ and L by

$$\|\hat{L}\| = \sup\{|\hat{L}(x)| : \|x\| \leqslant 1\},$$
$$\|L\| = \sup\{|L(x_1, \dots, x_m)| : \|x_i\| \leqslant 1 \ (i = 1, \dots, m)\}.$$

Mazur and Orlicz investigated relationships between $\|L\|$ and $\|\hat{L}\|$, and in the Scottish Book **[8]** conjectured that for any normed space E

$$K(m, E) \leqslant m^m/m!,$$

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where

$$K(m, E) = \min\{M: \|L\| \leqslant M\|\hat{L}\| \text{ for every } L \in \mathscr{L}_m^s(E, K)\}.$$

(We shall write $R(m, E)$, $C(m, E)$ instead of $K(m, E)$ if the normed space is real, complex respectively.) This conjecture was subsequently proved by Martin [4]. Notice that the constant $m^m/m!$ depends only on the integer m and not on the normed space.

Let $\Phi \in \mathscr{L}_m^s(l_m^1, K)$ be defined by

$$m!\Phi(x^1, \dots, x^m) = \sum_{\sigma \in S_m} x_1^{\sigma(1)} \cdots x_m^{\sigma(m)}, \tag{1}$$

where $x^i = (x_n^i)_{n=1}^m$ for $i = 1, \dots, m$ and S_m is the set of permutations of the first m-natural numbers. It can be shown, see [1, p. 45], that for this special Φ we have

$$\|\Phi\| = (m^m/m!)\|\hat{\Phi}\|.$$

Hence the universal constant $m^m/m!$ is the best possible. In the following, E denotes a complex normed space, unless otherwise specified. The distance between two Banach spaces X and Y is defined by

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\|: T \text{ is a linear isomorphism from } X \text{ onto } Y\}.$$

We say that X contains an almost isometric copy of Y if for any $\varepsilon > 0$, there exists a subspace Z of X such that $d(Y, Z) < 1 + \varepsilon$. (In other words Y is $(1 + \varepsilon)$-isomorphic to Z.)

We now come to our first main result.

THEOREM 1. *Suppose that $C(m, E) = m^m/m!$ for some positive integer m. Then E contains an almost isometric copy of l_m^1.*

To prove Theorem 1 we need a polarization formula. We define the function $s_{n,\beta}$ on $[0, 1]$ by

$$s_{n,\beta}(t) = e^{i\beta_n} r_n(t), \qquad n = 1, 2, \dots,$$

where $\beta = (\beta_1, \beta_2, \dots)$ is a sequence of real numbers and r_n is the nth Rademacher function. The functions $\{s_{n,\beta}\}_{n=1}^\infty$ form an orthonormal set in $L^2([0, 1], dt)$, where dt denotes Lebesgue measure. We omit the proof of the following lemma which is similar to the proof of Lemma 2 in [5].

LEMMA 1 (POLARIZATION FORMULA). *If E is a complex vector space, if L is a symmetric m-linear form, and if $x_1, \dots, x_m$ belong to E, then for any $\beta = (\beta_1, \dots, \beta_m)$*

(2)

$$m!L(x_1, \dots, x_m) = e^{-2i(\beta_1 + \cdots + \beta_m)} \int_0^1 s_{1,\beta}(t) \cdots s_{m,\beta}(t) \hat{L}\left(s_{1,\beta}(t)x_1 + \cdots + s_{m,\beta}(t)x_m\right) dt.$$

PROOF OF THEOREM 1. For given $\varepsilon > 0$, $0 < \varepsilon < 1$, set

$$\varepsilon_1 = (m^m/2^m)\left[1 - (1 - \varepsilon/2m)^m\right],$$

where m is a positive integer. Since $C(m, E) = m^m/m!$ there exists an $L \in \mathscr{L}_m^s(E, \mathbf{C})$ and unit vectors $x_1, \ldots, x_m$ in E such that

$$(3) \qquad |L(x_1, \ldots, x_m)| \geqslant ((m^m - \varepsilon_1)/m!)\|\hat{L}\|.$$

If $a_1, \ldots, a_m$ are any complex numbers, we can assume that

$$|a_1| = \max\{|a_i|: i = 1, \ldots, m\}.$$

Suppose that $a_k = |a_k|e^{i\alpha_k}$, $k = 1, \ldots, m$, and put $\beta_j = \alpha_j - \alpha_1$, $j = 2, \ldots, m$. Then

$$\begin{aligned}
\sum_{k=1}^{m} |a_k| &\geqslant \|a_1x_1 + a_2x_2 + \cdots + a_mx_m\| \\
&= \big\||a_1|x_1 + |a_2|e^{i\beta_2}x_2 + \cdots + |a_m|e^{i\beta_m}x_m\big\| \\
&= \big\||a_1|(x_1 + e^{i\beta_2}x_2 + \cdots + e^{i\beta_m}x_m) + e^{i\beta_2}x_2(|a_2| - |a_1|) \\
&\qquad + \cdots + e^{i\beta_m}x_m(|a_m| - |a_1|)\big\| \\
&\geqslant |a_1|\|x_1 + e^{i\beta_2}x_2 + \cdots + e^{i\beta_m}x_m\| + \sum_{k=2}^{m} |a_k| - (m-1)|a_1|.
\end{aligned}$$

So if we can prove that

$$(4) \qquad \|x_1 + e^{i\beta_2}x_2 + \cdots + e^{i\beta_m}x_m\| \geqslant m - \varepsilon/2$$

we shall get

$$(1 - \varepsilon/2)\sum_{k=1}^{m} |a_k| \leqslant \|a_1x_1 + a_2x_2 + \cdots + a_mx_m\| \leqslant \sum_{k=1}^{m} |a_k|.$$

Thus span $\{x_1, \ldots, x_m\}$ will be $(1 + \varepsilon)$-isomorphic to l_m^1 and the theorem will follow.

To prove (4) observe that (3) and the polarization formula (2), with $\beta_1 = 0$, imply

$$\sum_{\varepsilon_i = \pm 1} \|\varepsilon_1x_1 + \varepsilon_2e^{i\beta_2}x_2 + \cdots + \varepsilon_me^{i\beta_m}x_m\|^m \geqslant (m^m - \varepsilon_1)2^m.$$

Now from this last inequality we have

$$\|x_1 + e^{i\beta_2}x_2 + \cdots + e^{i\beta_m}x_m\|^m + (2^m - 1)m^m \geqslant (m^m - \varepsilon_1)2^m$$

and this proves (4).

From Theorem 1 we conclude that if $C(m, E) = m^m/m!$ for every m, then E contains uniformly isomorphic copies of l_m^1 for all m. So if E is a *Banach space*, then E has no type $p > 1$, since if it did (see [**6**]) it could not contain uniformly isomorphic copies of l_m^1 for all m. (For the definition of type p, see [**3**, p. 72].)

Notice that the condition that E should contain an almost isometric copy of l_m^1 does not always imply that $C(m, E) = m^m/m!$. To see this consider the complex Banach space l^∞. This is the space of all bounded complex-valued sequences $x = (x_i)$ under the norm

$$\|x\|_\infty = \sup\{|x_i|: i \in \mathbf{N}\}.$$

We know [**3**, p. 73] that l^∞ is not of type p for any $p > 1$ and so by Lemma 1.e.4 of [**3**], l^∞ contains almost isometric copies of l^1_m for every m. However

$$C(m, l^\infty) \leqslant m^{m/2}(m+1)^{(m+1)/2}/2^m m! < m^m/m!.$$

This was established by Harris [**2**, p. 154], see also [**7**].

Our second main result concerns norm-achieving extremal multilinear forms.

When E is a Banach space with $C(m, E) = m^m/m!$ we say that $L \in \mathscr{L}^s_m(E, \mathbf{C})$ is *extremal* if $\|L\| = (m^m/m!)\|\hat{L}\|$. We shall show that the only examples of such extremal L which achieve their norm are suitable "extensions" of the canonical example (1).

Given a positive integer m, let $n_1, \ldots, n_k$ be nonnegative integers with $n_1 + \cdots + n_k = m$. If $L \in \mathscr{L}^s_m(E, \mathbf{C})$, we write $L(x_1^{n_1} \cdots x_k^{n_k})$ for $L(x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k)$, where x_1 appears n_1 times, x_2 appears n_2 times, and so on.

THEOREM 2. *Let E be a Banach space and let $L \in \mathscr{L}^s_m(E, \mathbf{C})$ satisfy $\|L\| = (m^m/m!)\|\hat{L}\|$. If L achieves its norm at $(x_1, \ldots, x_m) \in E^m$, where $x_1, \ldots, x_m$ are unit vectors in E, then*

(a) *$\hat{L}$ achieves its norm, and*

(b) *$\hat{L}(x_1) = \cdots = \hat{L}(x_m) = 0$.*

This theorem is an immediate consequence of the following more general result.

THEOREM 2′. *Let E be a Banach space and let $L \in \mathscr{L}^s_m(E, \mathbf{C})$ satisfy $\|L\| = (m^m/m!)\|\hat{L}\|$. Then the following are equivalent:*

(i) *L achieves its norm at $(x_1, \ldots, x_m) \in E^m$, where $x_1, \ldots, x_m$ are unit vectors in E.*

(ii)(a) *$\hat{L}$ achieves its norm at the points $(e^{i\theta_1}x_1 + \cdots + e^{i\theta_m}x_m)/m$ for all choices of real numbers $\theta_1, \ldots, \theta_m$, and*

(b) *$L(x_1^{n_1} \cdots x_m^{n_m}) = 0$ for all m-tuples $(n_1, \ldots, n_m)$ of nonnegative integers, at least one of which is greater than 1, satisfying $n_1 + \cdots + n_m = m$.*

(iii)(a) *$\hat{L}$ achieves its norm at the point $(e^{i\theta_1}x_1 + \cdots + e^{i\theta_m}x_m)/m$ for some choice of real numbers $\theta_1, \ldots, \theta_m$, and*

(b) *$L(x_1^{n_1} \cdots x_m^{n_m}) = 0$ for all m-tuples $(n_1, \ldots, n_m)$ of nonnegative integers, at least one of which is greater than 1, satisfying $n_1 + \cdots + n_m = m$.*

PROOF. If (i) holds, then

$$\|L\| = |L(x_1, \ldots, x_m)| = (m^m/m!)\|\hat{L}\|. \tag{5}$$

We will prove that (5) implies (ii). Let T^m be the m-fold product of the circle group and let λ be Haar measure on T^m. Thus $d\lambda(\theta) = (1/2\pi)^m d\theta_1 \cdots d\theta_m$ and we can show easily that the following polarization formula holds:

$$m! L(x_1, \ldots, x_m) = \int_{T^m} e^{-i\theta_1} \cdots e^{-i\theta_m} \hat{L}\left(\sum_{j=1}^m x_j e^{i\theta_j}\right) d\lambda(\theta). \tag{6}$$

Now from (5) and (6) we get

$$\begin{aligned}(m^m/m!)\|\hat{L}\| &= \|L\| = |L(x_1,\dots,x_m)| \\ &= (1/m!)\left|\int_{T^m} e^{-i\theta_1}\cdots e^{-i\theta_m}\hat{L}\left(\sum_{j=1}^m x_j e^{i\theta_j}\right)d\lambda(\theta)\right| \\ &\leqslant (1/m!)\int_{T^m}\left|\hat{L}\left(\sum_{j=1}^m x_j e^{i\theta_j}\right)\right|d\lambda(\theta) \\ &\leqslant (1/m!)\|\hat{L}\|\int_{T^m}\left\|\sum_{j=1}^m x_j e^{i\theta_j}\right\|^m d\lambda(\theta) \leqslant (m^m/m!)\|\hat{L}\|.\end{aligned}$$

So we have

$$\left|\hat{L}\left(\sum_{j=1}^m x_j e^{i\theta_j}\right)\right| = \|\hat{L}\|\left\|\sum_{j=1}^m x_j e^{i\theta_j}\right\|^m,$$

$$\|e^{i\theta_1}x_1 + \cdots + e^{i\theta_m}x_m\| = m \tag{7}$$

for all choices of real numbers $\theta_1,\dots,\theta_m$. Thus

$$|\hat{L}((e^{i\theta_1}x_1 + \cdots + e^{i\theta_m}x_m)/m)| = \|\hat{L}\|$$

for all real numbers $\theta_1,\dots,\theta_m$ and so part (a) is proved. To prove part (b) note first of all that from the multinomial formula (see [**1**, p. 38]) we have

$$\begin{aligned}(m!/m^m)|L(x_1,\dots,x_m)| = \|\hat{L}\| &= \left|\hat{L}\left(\left(\sum_{j=1}^m e^{i\theta_j}x_j\right)/m\right)\right| \\ &= (1/m^m)\left|\sum(m!/n_1!\cdots n_m!)L\left((e^{i\theta_1}x_1)^{n_1}\cdots(e^{i\theta_m}x_m)^{n_m}\right)\right|,\end{aligned}$$

where the summation is over all m-tuples $(n_1,\dots,n_m)$ of nonnegative integers satisfying $n_1 + \cdots + n_m = m$. Since the last equation is true for all real numbers $\theta_1,\dots,\theta_m$ we get

$$\begin{aligned}&(m!|L(x_1,\dots,x_m)|)^2 \\ &\quad = \int_{T^m}\left|\sum(m!/n_1!\cdots n_m!)L\left((e^{i\theta_1}x_1)^{n_1}\cdots(e^{i\theta_m}x_m)^{n_m}\right)\right|^2 d\lambda(\theta).\end{aligned}$$

From the above equation it follows that

$$\sum\left((m!/n_1!\cdots n_m!)|L(x_1^{n_1}\cdots x_m^{n_m})|\right)^2 = 0,$$

where the summation is over all m-tuples $(n_1,\dots,n_m)$ of nonnegative integers, at least one of which is greater than 1, satisfying $n_1 + \cdots + n_m = m$. This proves part (b).

Since (ii) obviously implies (iii) we have to prove only that (iii) implies (i). But conditions (a), (b) of (iii) and the multinomial formula give us

$$\|\hat{L}\| = (m!/m^m)|L(x_1,\dots,x_m)|$$

and since by hypothesis we have $\|L\| = (m^m/m!)\|\hat{L}\|$, it follows that

$$\|L\| = |L(x_1, \ldots, x_m)|.$$

Working in a similar fashion, it can also be proved that if E is a Banach space and $|L(x_1, \ldots, x_m)| = \|L\|$ for some $L \in \mathscr{L}_m^s(E, \mathbf{C})$, where $x_1, \ldots, x_m$ are unit vectors in E, then conditions (ii) and (iii) of Theorem 2′ are equivalent to the condition

(i′)

$$\|L\| = (m^m/m!)\|\hat{L}\|.$$

The following example shows that the converse of Theorem 2 is false, so for equivalent conditions the complications of Theorem 2′ are necessary.

EXAMPLE. We consider the canonical example (1) in the case $m = 3$. Then

$$\|\Phi\| = (3^3/3!)\|\hat{\Phi}\|.$$

Also for the unit vectors $x = (1/2, 1/2, 0)$, $y = (1/2, 0, 1/2)$, $z = (0, 1/2, 1/2)$ of l_3^1 we have $\hat{\Phi}(x) = \hat{\Phi}(y) = \hat{\Phi}(z) = 0$ and $\hat{\Phi}$ achieves its norm at the point $(1/3, 1/3, 1/3)$. However $\Phi(x, y, z) = 1/24 < \|\Phi\|$, since $\|\Phi\| = 1/6$.

Suppose that $L \in \mathscr{L}_m^s(E, \mathbf{C})$ satisfies (5) and let F be the restriction of L to $B = \operatorname{span}\{x_1, \ldots, x_m\}$. Since (7) holds, it follows from the proof of Theorem 1 that for any complex numbers $a_1, \ldots, a_m$ we have

$$\|a_1 x_1 + \cdots + a_m x_m\| = \sum_{k=1}^{m} |a_k|.$$

Thus B is isometrically isomorphic to l_m^1. Note also that

$$|F(x_1, \ldots, x_m)| = (m^m/m!)\|\hat{F}\|.$$

Now since F satisfies condition (ii)(b) we get the following result.

COROLLARY 1. *Let $C(m, E) = m^m/m!$ and L be an extremal continuous symmetric m-linear form, which achieves its norm at the point $x = (x_1, \ldots, x_m)$ of the unit sphere of E^m. Then $F = L|_B$, where $B = \operatorname{span}\{x_1, \ldots, x_m\}$ is a continuous symmetric m-linear form on B, and $F = c\Phi$, where c is a constant and $\Phi \in \mathscr{L}_m^s(l_m^1, \mathbf{C})$ is given by* (1).

If E is an m-dimensional Banach space, then every multilinear form on E is continuous. If L is a symmetric m-linear form on E, then L achieves its norm since the unit ball of E^m is compact. Using these remarks we obtain another corollary.

COROLLARY 2. *An m-dimensional Banach space E is isometrically isomorphic to l_m^1 if and only if $C(m, E) = m^m/m!$. If $C(m, E) = m^m/m!$, then every extremal symmetric m-linear form on E is of the form $L = c\Phi$ for some constant c and $\Phi \in \mathscr{L}_m^s(l_m^1, \mathbf{C})$ defined by* (1).

Finally notice that using techniques similar to those of the proof of Theorem 1, we can prove Theorem 1 in the case where E is a real normed space. However for Theorem 2′, if E is a real Banach space, we need a different approach. We hope to discuss this in a subsequent paper.

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