METRICS OF NEGATIVE CURVATURE
ON VECTOR BUNDLES
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ABSTRACT. It is shown that any vector bundle $E$ over a compact base manifold $M$ admits a complete metric of negative (respectively nonpositive) curvature provided $M$ admits a metric of negative (nonpositive) curvature.

1. Introduction. The purpose of this note is to prove the following

THEOREM. Let $B$ be a compact $n$-dimensional manifold of negative sectional curvature. Then any vector bundle $\Pi: E \to B$ admits a complete metric of negative sectional curvature $K_E$ satisfying $-a \leq K_E \leq -1$ for some constant $a \geq 1$. (Here $a$ depends on the geometry of $B$ and the topology of the bundle $\Pi: E \to B$.)

If $B$ is a compact manifold of nonpositive sectional curvature, then any vector bundle $\Pi: E \to B$ admits a complete metric of nonpositive sectional curvature $K_E$ satisfying $-b \leq K \leq 0$ for some positive constant $b$.

This result should be compared with a well-known open problem of Gromoll: If $M$ is a compact manifold of positive sectional curvature, does every vector bundle over $M$ admit a complete metric of nonnegative sectional curvature?

The theorem was motivated by, and partially answers, a question of M. Gromov: Does every vector bundle over a compact base $B$, with a possibly singular metric of negative curvature on $B$, admit a smooth complete metric of negative curvature (cf. [3] for a discussion of a singular metrics). For example, let $T$ be a hyperbolic group, in the sense of [2], and let $X$ be a metric space on which $T$ acts freely with compact quotient. One may ask if there is an embedding of $X$ in $\mathbb{R}^n$ such that a tubular neighborhood of $X \subset \mathbb{R}^n$ admits a complete metric of negative sectional curvature. This approach is relevant for the Novikov conjecture for such hyperbolic groups.

It is of interest to note that Gromov, Lawson and Thurston [4] have recently shown that most 2-plane bundles $E$ over a compact Riemann surface $M_g$, of genus greater than one, admit complete metrics of constant curvature $-1$, provided $|\chi(E)| \leq |\chi(M_g)|$.

I am grateful to M. Gromov for suggesting this problem and for interesting discussions.

2. Preliminaries. We begin with the standard topological description of vector bundles. Let $\Pi_0: P \to B$ be a right principal $O(m)$ bundle, $m \geq 1$, over a smooth $n$-dimensional manifold $B$. Let $G = O(m)$ act on $\mathbb{R}^m$ on the left in the usual
way by orthogonal transformations. Define an action of $G$ on $P \times \mathbb{R}^m$ by $g(p, f) = (pg, g^{-1}f)$. Then the quotient space $E = P \times \mathbb{R}^m/G$ is a vector bundle $\Pi_B : E \to B$ with fiber $F$ diffeomorphic to $\mathbb{R}^m$ and structure group $G$. $E$ is called the vector bundle associated to $P$. Conversely, given a vector bundle $V$ over $B$, we may assume without loss of generality that its structure group is $O(m)$. Then there is a principal $O(m)$ bundle over $B$ such that associated bundle constructed above is equivalent to $V$.

Let $\langle \cdot, \cdot \rangle_G$ denote the negative of the killing form of the Lie algebra $L(G)$ of $G$; we will also let $\langle \cdot, \cdot \rangle_G$ denote the corresponding bi-invariant metric on $G$. Let $\langle \cdot, \cdot \rangle_B = ds^2_B$ denote a smooth Riemannian metric on $B$. If $\Theta : TP \to L(G)$ is any connection 1-form on $P$, we define a metric on $P$ by

$$ds^2_P = \Pi^*_0(ds^2_B) + \Theta \cdot \Theta,$$

e.q. for vectors $x, y \in TP, \langle x, y \rangle_P = \langle \Pi^*_0 x, \Pi^*_0 y \rangle_B + \langle \Theta(x), \Theta(y) \rangle_G.$

It is well known (cf. [5]) that $\Pi_0 : P \to B$ is a Riemannian submersion, with totally geodesic fibers, with respect to the metrics $ds^2_P$ and $ds^2_B$. Let $H^1$ denote the orthogonal complement of the tangent space to the orbits $G \subset P$. Then $H^1$ coincides with the horizontal spaces for the submersion $\Pi_0$, as well as the horizontal spaces for the connection 1-form.

Next we consider the product metric

$$ds^2_{P \times F} = ds^2_P + ds^2_F$$
on $P \times F$, where $ds^2_F$ is the metric of constant curvature $-a^2$ on $F \approx \mathbb{R}^m$; of course $a = 0$ if $m = 1$. Note that $ds^2_{P \times F}$ is invariant under the action of $G$ on $P \times F$, so that $ds^2_{P \times F}$ descends to give a metric $ds^2_E$ on $E$. We have the following commutative diagram:

\[
\begin{array}{ccc}
G & \to & G \times F \\
\downarrow & & \downarrow \\
P \times F & \to & E \\
\Pi \downarrow & & \downarrow \\
B & \leftarrow & F
\end{array}
\]

With respect to the metrics defined above, each map $\Pi, \Pi_1, \Pi_B$ is a Riemannian submersion with totally geodesic fibers (cf. [5] for a proof).

For later purposes, we recall a formula of O'Neiill [6] relating the curvature of the base and total space of Riemannian submersion. Let $S \to M$ be a Riemannian submersion. Let $X, Y$ be horizontal vector fields on $S$ and let $X_* = \Pi_* X, Y_* = \Pi_* Y$. Then if $K$ denotes sectional curvature, we have

$$K^S(X, Y) = K^M(X_*, Y_*) - \frac{3}{4} \frac{\|X, Y\|_{\gamma}^2}{\|X \wedge Y\|^2},$$

where $[X, Y]_\gamma$ denotes the orthogonal projection of the Lie bracket $[X, Y]$ onto the vertical subspaces of $T(S)$.

3. Construction of metrics. The metric $ds^2_E$ constructed in §2 does not have negative sectional curvature. In fact, the O'Neill formulas [6] imply that the
"mixed" curvature $K^E(X, V)$ for $X$ horizontal and $V$ vertical with respect to $\Pi_B$ are nonnegative.

In order to construct metrics of negative curvature on $E$, we consider warped product metrics on $P \times F$. Let $g: F \rightarrow \mathbb{R}$ be an $O(m)$-invariant smooth function, with $g > 0$. Thus, $g = g(r)$, where $r$ is the distance function to $0 \in F$ with respect to the metric $ds^2_F$. We will specify $g$ more precisely later in this section. Extend $g$ to a function $g: P \times F \rightarrow \mathbb{R}$ by first projecting on the second factor. We consider metrics of the form

$$\text{(3.1)} \quad ds^2_{P \times F} = g^2 \cdot ds^2_P + ds^2_F.$$ 

Note that $ds^2$ is also a $G$-invariant metric and so gives a metric $\tilde{ds}^2_E$ on $E$. The projection $\Pi: P \times F \rightarrow E$ is a Riemannian submersion with respect to these metrics; the fibers are no longer totally geodesic however. Nevertheless, one may still use (2.3) to relate the curvatures.

We will need explicit descriptions of the horizontal and vertical spaces of $\Pi$ in these metrics. Thus, let $X(M)$ denote the space of $C^\infty$ vector fields on $M$. Define maps

$$L(G) \rightarrow X(P), \quad E \rightarrow \tilde{E}(p), \quad \text{and} \quad L(G) \rightarrow X(F), \quad E \rightarrow \tilde{E}(f),$$

where

$$\text{(3.2)} \quad \tilde{E}(p) = \frac{d}{dt} (p \cdot \exp tE) \bigg|_{t=0}, \quad \tilde{E}(f) = \frac{d}{dt} (\exp tE \cdot f) \bigg|_{t=0}.$$

It is well known, and easy to verify, that these maps are Lie algebra homomorphisms. We note there is a constant $C > 0$ such that

$$\text{(3.3)} \quad \frac{1}{C} < \frac{\|\tilde{E}(p)\|}{|E|} < C$$

for all $p \in F$, for any given smooth metric on $P$. For any $f \in F$, we may choose $(m - 1)$ unit vectors $e_i \in L(G)$, depending on $f$, such that $\{\tilde{e}_i(f)/\psi(r)\}_{i=1}^{m-1}$ is an orthonormal basis of $T_fS \subset T_fF$, where $S$ is the geodesic $r$-sphere through $f$ centered at $0$. One calculates that

$$\text{(3.4)} \quad \psi(r) = \frac{1}{a} \sinh ar.$$

Thus $\{\tilde{e}_i(f)/\psi(r), \nabla r\}$ forms an orthonormal basis of $T_pF$. Note that $\tilde{E}(f) = 0$ for any $E \notin \text{span}\{e_i\}_{i=1}^{m-1}$.

One easily sees that the vertical space $V_{p,f} \subset T_{(p,f)}P \times F$ for $\Pi$ is given by

$$V_{p,f} = \text{span}_{E \in L(G)} [\tilde{E}(p) - \tilde{E}(f)].$$

By the remarks above, we may choose a basis $\{e_i\} \in L(G)$, depending on $f$, such that

$$V_{p,f} = \text{span}_{i=1}^{m-1} [\tilde{e}_i(p) - \tilde{e}_i(f)] \oplus \text{span}_{i=m}^{N} [\tilde{e}_i(p)],$$

where $N = \dim G$. Note that $\dim V_{p,f} = N$. Let $H^1_{p,f} = (T_pG)^\perp \subset T_pP$ as in §2, $H^2 = \text{span}_{i=1}^{m-1} [\alpha \tilde{e}_i(p) + \tilde{e}_i(f)]$, where

$$\alpha(p, f) = \frac{1}{g^2(p,f)} \frac{\langle \tilde{E}(f), \tilde{E}(f) \rangle}{\langle \tilde{E}(p), \tilde{E}(p) \rangle},$$

and let $H^3 = \nabla r$. 

Then there is an orthogonal splitting, with respect to \( ds^3_{P \times F} \), of the form

\[
T(P \times F) = V \oplus H^1 \oplus H^2 \oplus H^3.
\]

The subspace \( H^1 \oplus H^2 \oplus H^3 \) is the horizontal space for the submersion \( P \times F \to E \) with respect to the metrics \( ds^2_{P \times F} \) and \( ds^2_E \).

We now begin with the computation of the curvature of \( ds^3_E \). First, by (2.3), the curvature of \((P \times F, ds^2_{P \times F}) \) and \((E, ds^2_E) \) are related by

\[
\tilde{K}^E(X_*, Y_*) = \tilde{K}^{P \times F}(X, Y) + \frac{3}{4} \frac{|[X, Y]|^2}{|X \wedge Y|}.
\]

for horizontal vectors \( X, Y \in T(P \times F) \). To estimate the first term, we use the formula for the sectional curvature of a warped product given in [1]. Write \( X = X_p + X_F \), where \( X_p \) (resp. \( X_F \)) is the orthogonal projection of \( X \) onto \( TP \) (resp. \( TF \)). If the pair \( \{X, Y\} \) is orthonormal with respect to \( ds^2 \), then

\[
\tilde{K}^{P \times F}(X, Y) = K^F(X_F, Y_F) \cdot |X_F \wedge Y_F|^2 - g(|Y_p|^2D^2g(X_F, X_F) - 2(X_p, Y_p))
\]

\[
+ g^2[K^P(X_p, Y_p) - \frac{1}{g^2}][X_p \wedge Y_p| |X_F \wedge Y_F|^2
\]

Let \( B_\varepsilon \) denote the geodesic ball of radius \( \varepsilon \) about \( 0 \in F \). The function \( g \) will depend on a parameter \( \varepsilon \), to be fixed below, and chosen to satisfy the following properties:

(i) \( g \) is convex, i.e. \( D^2g \geq 0 \) on \( F \) and \( D^2g < C_0g \) outside \( B_\varepsilon \).

(ii) \( |\nabla g|^2/g^2 > C_1 \) outside \( B_\varepsilon \).

(iii) \( |\nabla g|^2/g^2 < C_2 \) outside some compact set of \( F \).

(iv) \( |\nabla g|^2(x) > C_3 \cdot r(x) \) for \( x \in B_\varepsilon \).

(v) \( g \leq 1 \) in \( B_\varepsilon \), \( g > 1 \) outside \( B_\varepsilon \).

Here \( C_0, C_1, C_2, C_3 \) are constants, also to be specified below. For example, one may choose \( g \) of the form

\[
g = \{a_1 + a_2r^{3/2}\}e^{asr}
\]

and adjust \( \{a_i\} \) to satisfy (3.8). Basically, \( a_1 \) is small and \( a_2, a_3 \) large.

Using (3.8) we may estimate (3.7). First, since \( g \) is convex, the second term in (3.7) within the brackets is nonnegative. Since \( F \) has curvature \( -a^2 \), we find

\[
\tilde{K}^{P \times F}(X, Y) \leq -a^2|X_F \wedge Y_F|^2 + 2g[K^P(X_p, Y_p) - |\nabla g|^2]|X_p \wedge Y_p|^2.
\]

We now consider several cases. Suppose \( f \notin B_\varepsilon \). Choose \( C_1 = a^2 + \sup K^P(X_p, Y_p) \).

By (3.8)(ii) and (v) we obtain

\[
\tilde{K}^{P \times F}(X, Y) \leq -a^2|X_F \wedge Y_F|^2 - a^2g^2|X_p \wedge Y_p|^2
\]

\[
\leq a^2|X_F \wedge Y_F|^2 - a^2|X_p \wedge Y_p|^2 \leq -a^2/4.
\]

Next suppose \( f \in B_\varepsilon \). If \( |X_F \wedge Y_F|^2 \geq |X_p \wedge Y_p|^2 \), then setting \( b = \sup K^P(X_p, Y_p) \) we have

\[
\tilde{K}^{P \times F}(X, Y) \leq -a^2|X_F \wedge Y_F|^2 + b|X_p \wedge Y_p|^2
\]

\[
\leq \left[ -a^2 + \frac{b}{g^2} \right]|X_F \wedge Y_F|^2 \leq \frac{1}{4} \left[ -a^2 + \frac{b}{g^2} \right]
\]

assuming \( -a^2 + b/g^2 \leq 0 \).
Finally, suppose $|X_p \wedge Y_p|^2 < |X_p \wedge Y_p|^2$ and $f \in B_{\varepsilon}$. We may write $X_p = X_B + X_2$, where $X_B \in H^1$ and $X_2 = \alpha \sum_{i=1}^{n-1} a_i e_i(p) \in (H^2)_p$. It is important to note that $|X_2| \sim -\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. To see this, we have $|X_F| < 1$, so that $|\sum a_i e_i(f)| < 1$. Since, by definition, $\alpha = O(|e(f)|^2)$, the claim follows. Note also that $|X_B| \sim$ is bounded away from zero as $f \rightarrow 0$, since by our assumption $|X_B|$ is bounded away from zero. These same remarks apply to $Y_p$ and we obtain the estimate

$$K^P(X_p, Y_p) = K^P(X_B, Y_B) + O(\varepsilon).$$

Now (2.3) applied to the Riemannian submersion $\Pi_0: P \rightarrow B$ gives

$$K^P(X_B, Y_B) = K^B(X_B, Y_B) - \frac{3}{4} \frac{|[X_B, Y_B]|^2}{|X_B \wedge Y_B|^2}.$$

Thus, for the last case, we obtain

$$\begin{align*}
\tilde{K}^P \times F(X, Y) &\leq -a^2 |X_F \wedge Y_F|^2 \\
&+ \left[ K^B(X_B, Y_B) - \frac{3}{4} \frac{|[X_B, Y_B]|^2}{|X_B \wedge Y_B|^2} - |\nabla g|^2 + O(\varepsilon) \right] g^2 |X_P \wedge Y|^2.
\end{align*}$$

In order to estimate the second term of (3.6), we use the following Lemma.

**Lemma.** Let $X, Y$ be horizontal fields on $(P \times F, ds^2)$. Then there is a constant $k$, depending on $ds^2_p$ and inf $g$, but not on $a$, such that

$$||[X, Y]||^2 < k \cdot |X \wedge Y|^2.$$

**Proof.** Since both sides of (3.12) are bilinear, it is sufficient to check (3.12) on a basis for the horizontal fields. Thus, let $X = \sum X_i, Y = \sum Y_i$, where $X_i, Y_i \in H^i$. One verifies that

$$[X_3, Y_1] = [X_2, Y_3] = 0, \quad [X_1, Y_2] = [X_2, Y_1] = 0.$$ 

Thus $[X, Y] = [X_1, Y_1] + [X_2, Y_2]$ and

$$||[X, Y]||^2 < ||[X_1, Y_1]||^2 + ||[X_2, Y_2]||^2.$$

Applying (2.3) to the submersion $\Pi_0: P \rightarrow B$ gives

$$K^P(X_1, Y_1) = K^B(X_1, Y_1) - \frac{3}{4} \frac{|[X_1, Y_1]|^2}{|X_1 \wedge Y_1|^2},$$

note that since $X_1, Y_1$, and $[X_1, Y_1] \in TP$, the vertical projections for $\Pi_0$ and $\Pi$ agree. Since $K^P$ and $K^B$ are bounded, we have

$$||[X_1, Y_1]||^2 < k |X_1 \wedge Y_1|^2,$$

and thus

$$||[X_1, Y_1]||^2 < k |X_1 \wedge Y_1|^2.$$

We estimate the second term in (3.13) on a basis of the form $B_i = \alpha e_i(p) + e_i(f)$, where at a given $p_0 \in P$, we assume $(B_i, B_j)(p_0, f) = 0$ if $i \neq j$. We have

$$[B_i, B_j] = \alpha^2 [e_i(p), e_j(p)] + [e_i(f), e_j(f)] = \alpha^2 C_{ij}^k e_k(p) + C_{ij}^k e_k(f),$$

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where we have used the fact that the maps $E \to \tilde{E}(p)$ and $E \to \tilde{E}(f)$ are Lie algebra homomorphisms; here $C^k_{ij}$ are the structure constants of $L(G)$. Thus

$$
|[B_i, B_j]|^2 \sim \sum_{k,l,m} \frac{[\alpha^2 C^k_{ij}(e_k(p), e_k(m)) - C^k_{ij}(e_l(f), e_l(m))]|e_m(p) - e_l(f)|^2}{|e_m(p) - e_l(f)|^2} \leq C \cdot \frac{\alpha^4 |e(p)|^4 + |e(f)|^4}{|e(p) - e(f)|^2},
$$

where $C$ is a constant independent of the metrics. Since $|e(p) - e(f)|^2 \geq L$ for some constant $L$ depending only on $\inf g$, we have

$$
|[B_i, B_j]|^2 \leq C^1[\alpha^4 |e(p)|^4 + |e(f)|^4].
$$

On the other hand,

$$
|B_i \wedge B_j|^2 = |B_i|^2 |B_j|^2 - <B_i, B_j>|^2 = |\alpha e_i(p) + e_i(f)|^2 - |\alpha e_j(p) + e_j(f)|^2 \leq C[\alpha^4 |e(p)|^4 + |e(f)|^4].
$$

Combining the last two estimates with (3.15) gives the result.

We now combine the above estimates to determine $K^E(X, Y)$. As before, we deal with several cases. We assume $m \geq 2$ and will discuss the case $m = 1$ at the end

(i) $f \not\in B_\varepsilon$: Combining (3.9) and (3.12) and substituting into (3.6) gives

$$
(3.16) \quad K^E(X, Y) \leq -\frac{a^2}{4} + k.
$$

(ii) $f \in B_\varepsilon$ and $|X_F \wedge Y_F|^2 \geq |X_p \wedge Y_p|^2$: Using (3.10) and (3.12) as above gives

$$
(3.17) \quad K^E(X, Y) \leq \frac{1}{4}[-a^2 + b/g^2] + k.
$$

Thus, making a choice of $g$ satisfying (3.8), we see that we may choose a sufficiently large so that $K^E(X, Y) < 0$ in the above two cases. In particular, the curvature of $E$ may be made negative outside a neighborhood of the 0-section of $\Pi_B: E \to B$, regardless of the curvature of $B$.

(iii) $f \in B_\varepsilon$ and $|X_F \wedge Y_F|^2 < |X_p \wedge Y_p|^2$: Using (3.11) and the fact that $|X \wedge Y| = 1$, we estimate (3.6) as

$$
(3.18) \quad K^E(X, Y) \leq -\frac{a^2}{4} |X_F \wedge Y_F|^2 + \frac{1}{4} |X_p \wedge Y_p|^2 + \frac{3}{4} |X_1, Y_1|^2 + \frac{3}{4} |X_2, Y_2|^2 - |\nabla g|^2 + O(\varepsilon)
$$

$$
\leq -\frac{a^2}{4} |X_F \wedge Y_F|^2 + |K^B(X_B, Y_B) + O(\varepsilon) - |\nabla g|^2| \frac{1}{g^2} |X_p \wedge Y_p|^2 + O(\varepsilon),
$$

where we have used (3.8)(v).

Now suppose first that $K^B(X_B, Y_B) < 0$, say $K^B(X_B, Y_B) \leq -m^2 < 0$. Choosing $\varepsilon$ sufficiently small in (3.18), we obtain

$$
(3.19) \quad K^E(X, Y) \leq -C
$$
for some constant \( C > 0 \). We may combine (3.19) with (3.16) and (3.17) and rescale the metric if necessary to obtain \( \tilde{K}^E(X, Y) \leq -1 \) for all \( X, Y \in T(E) \).

Next suppose only \( K^B(X_B, Y_B) \leq 0 \). Then by (3.18)

\[
\tilde{K}^E(X, Y) \leq -a^2 |X_F \wedge Y_F|^2 + |O(\varepsilon) - C_3 \varepsilon| \frac{1}{g^2} |X_p \wedge Y_p|^2 + O(\varepsilon).
\]

We may choose \( \varepsilon \) sufficiently small and \( C_3 \) sufficiently large in (3.8) (iv) so that \( |O(\varepsilon) - C_3 \varepsilon| |X_p \wedge Y_p|^2 / g^2 \) is sufficiently negative for \( \varepsilon \neq 0 \), to dominate the last \( O(\varepsilon) \) term. We then obtain \( \tilde{K}^E(X, Y) \leq 0 \). Combining this with (3.16) and (3.17) gives a complete metric on \( E \) of nonpositive sectional curvature.

Finally, it is straightforward to verify that the condition \( D^2g < C_0g \) outside \( B_\varepsilon \) for some constant \( C_0 \) implies in both cases \( K^B < 0 \) and \( K^B \leq 0 \) that

\[
\tilde{K}^E(X, Y) \geq -M^2
\]

for some constant \( M \). This proves the theorem in the case \( m \geq 2 \).

Suppose finally that \( m = 1 \). In the notation above, any horizontal 2-plane for \( \Pi: P \times \mathbb{R} \rightarrow E \) has a basis of the form \( X = X_B + c \cdot \nabla r, Y = Y_B \). Since \([X, Y] V = 0\), we obtain from (3.7)

\[
\tilde{K}^E(X, Y) = \tilde{K}^{P \times \mathbb{R}}(X, Y) = -g c^2 |Y_B|^2 g^{11} + g^2 [K^B(X_B, Y_B) - |\nabla g|^2] |X_B \wedge Y_B|^2.
\]

This can be made negative, respectively nonpositive, depending on the curvature of \( B \), by choosing \( g \) to be any convex function. In particular, \( g \) satisfying (3.8) suffices to prove the theorem in this case also.

REFERENCES


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