Pin^c COBORDISM AND EQUIVARIANT Spin^c COBORDISM OF CYCLIC 2-GROUPS

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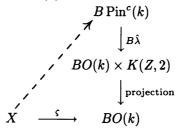
ABSTRACT. We compute the additive structure of $\Omega_*^{\mathrm{Pin}^c}$ and $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_{2^v})$ using the Anderson, Brown, and Peterson splitting of the spectrum $M\,\mathrm{Spin}^c$.

In this note, we determine the groups $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_{2^v})$ and $\Omega_*^{\mathrm{Pin}^c}$. This paper is purely topological and is companion to Bahri and Gilkey [3], where these groups are studied by analytic methods using the eta invariant. The present calculation is considerably shorter than our earlier Adams spectral sequence computation. Our approach is along lines suggested kindly by the referee and by Professor R. E. Stong and we acknowledge their help with gratitude. We also acknowledge with pleasure many helpful conversations with Professor M. Bendersky.

We begin by defining the groups Pin$^c(k)$. Let $O(k)$ denote the orthogonal group and (for $k > 2$) Pin(k) its universal covering group. The group Z_2 acts on Pin$(k) \times S^1$ by sending $(g, z) \in \mathrm{Pin}(k) \times S^1$ to $(-g, -z)$ and Pin$^c(k)$ is the quotient group Pin$(k) \times_{Z_2} S^1$. Let $\pi\colon \mathrm{Pin}(k) \to O(k)$ denote the covering projection and let $\lambda(g, z) = (\pi g, z^2)$ define a covering projection $\lambda\colon \mathrm{Pin}(k) \times S^1 \to O(k) \times S^1$. The map λ extends to a map $\hat{\lambda}\colon \mathrm{Pin}^c(k) \to O(k) \times S^1$ and defines a short exact sequence: $1 \to Z_2 \to \mathrm{Pin}^c(k) \xrightarrow[\hat{\lambda}]{} O(k) \times S^1 \to 1$. Associated to this exact sequence is a fibration:

$$B\,\mathrm{Pin}^c(k)$$
$$\Big\downarrow{\scriptstyle B\hat{\lambda}}$$
$$BO(k) \times K(Z, 2) \xrightarrow{\ g\ } K(Z_2; 2)$$

An argument along the lines of that outlined in Stong [6, p. 292] shows that g represents the cohomology class $w_2 \otimes 1 + 1 \otimes \iota$ in $H^2(BO(k) \times K(Z, 2))$, where w_2 is the second Stiefel-Whitney class and ι is the generator in $H^2(K(Z, 2))$. A real k-plane bundle ς over X is said to have a Pin^c structure if, in the following diagram its classifying map lifts to $B\,\mathrm{Pin}^c(k)$:

$$B\,\mathrm{Pin}^c(k)$$
$$\Big\downarrow{\scriptstyle B\hat{\lambda}}$$
$$BO(k) \times K(Z, 2)$$
$$\Big\downarrow{\scriptstyle \text{projection}}$$
$$X \xrightarrow{\ \varsigma\ } BO(k)$$

It follows that ς admits a Pin$^c(k)$ structure if and only if $w_2(\varsigma)$ is the reduction of an integral cohomology class. A manifold is said to be a Pin^c manifold if its stable normal bundle has a Pin^c structure. Such a structure is not unique and inequivalent Pin^c structures are parametrized by complex line bundles. There is an associated bordism theory $\Omega_*^{\mathrm{Pin}^c}$ which is not a ring. The group Spin$^c(k)$ is defined similarly; the group $O(k)$ is replaced by $SO(k)$ throughout. A Spin^c bundle ς also has $w_1(\varsigma) = 0$ and so $\Omega_*^{\mathrm{Spin}^c}$ is a ring.

Let $I = (i_1, i_2, \ldots, i_k)$ be a collection of integers $i_\nu \geq 1$ sorted into ascending order $i_\nu \leq i_{\nu+1}$. Let $n(I) = \sum_\nu i_\nu$ be the sum of this collection. Let $\underline{bu}$ be the spectrum for connective complex K-theory and let $\underline{K(Z_2)}$ be the mod 2 Eilenberg-MacLane spectrum. Anderson, Brown, and Peterson [2] proved that there exist integers $r(j)$ so

$$\underline{M\,\mathrm{Spin}^c} \approx \left\{ \bigvee_I \sum{}^{4n(I)} \underline{bu} \right\} \vee \left\{ \bigvee_j r(j) \sum{}^j \underline{K(Z_2)} \right\}$$

is a 2-primary equivalence of spectra. (We also refer to Stong [7, p. 319] for further details.) The space BZ_{2^v} is 2-primary and consequently

$$\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_{2^v}) \approx \left\{ \bigoplus_I \sum{}^{4n(I)} \underline{bu}_*(BZ_{2^v}) \right\} \oplus \left\{ \bigoplus_j r(j) \cdot \sum{}^j H^*(BZ_{2^v}; Z_2) \right\}.$$

Let $R_0(Z_{2^v})$ denote the augmentation ideal of the representation ring of Z_{2^v}. Let $A_{2k-1}(2^v) = R_0(Z_{2^v}/R_0(Z_{2^v})^{k+1}$ and let $A_{2k}(2^v) = 0$. The Gysin sequence in $\underline{bu}$ homology associated to the bundle $Z_{2^v} \to S^1 \to S^1$ imposes relations on $\underline{bu}_{2k-1}(BZ_{2^v})$ equivalent to the relations in $A_{2k}(2^v)$ so that $\underline{bu}_{2k-1}(BZ_{2^v}) \approx A_{2k-1}(2^v)$. Let $Z_n = \{\lambda \in C : \lambda^n = 1\}$ act without fixed points on $S^{2k+1} \subseteq C^{k+1}$ by complex multiplication. Atiyah [1] proved $A_{2k-1}(2^v) \approx \tilde{K}(S^{2k+1}/Z_{2^v})$. The K-theory groups of S^{2k+1}/Z_{2^v} were computed by Fujii et al. [4, Theorem 31] and N. Mahammed [5]. We combine our computations with their results to conclude:

THEOREM 1. *Let* $[*]$ *be the greatest integer function. Let* $1 \leq i \leq 2^{v-1}$ *and let* $s = [\log_2 i]$. *If* $i \leq k$, *let* $t(i, k, v) = v - s + [(k - i)/2^s]$; *if* $i > k$, *let* $t(i, k, v) = 0$. *Then* $A_{2k-1}(2^v) = \bigoplus_{i=1}^{2^{v-1}} Z/2^{t(i,k,v)} \cdot Z$. *As graded Abelian groups,*

$$\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_{2^v}) \approx Z[x_4, x_8, \ldots, x_{4k}, \ldots] \otimes A_*(2^v) \oplus \tilde{H}_*(BZ_{2^v}; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})).$$

We conclude by proving $\tilde{\Omega}_n^{\mathrm{Pin}^c}$ and $\tilde{\Omega}_{n+1}^{\mathrm{Spin}^c}(BZ_2)$ are isomorphic. We mimic the argument given in Stong [6, pp. 293 and 354] where a similar isomorphism between $\Omega_n^{\mathrm{Spin}^c}$ and $\tilde{\Omega}_{n+2}^{\mathrm{Spin}}(CP^\infty)$ is constructed. Let γ_k be the k-plane bundle over $B\,\mathrm{Spin}^c(k)$ and let λ be the canonical bundle over BZ_2. The second Stiefel-Whitney class of γ_k has an integral representative so $w_2(\gamma_k \oplus \lambda)$ also has an integral representative. This yields a lifting δ in the following diagram:

$$
\begin{array}{ccc}
 & & B\,\mathrm{Pin}^c(k+1) \\
 & \overset{\delta}{\nearrow} & \downarrow \\
B\,\mathrm{Spin}^c(k) \times BZ_2 \xrightarrow[\gamma_k \times \lambda]{} BO(k) \times BO(1) \xrightarrow[\oplus]{} BO(k+1)
\end{array}
$$

The homotopy long exact sequence associated to the fibrations $1 \to U(1) \to \mathrm{Spin}^c(k) \to SO(k) \to 1$ and $1 \to U(1) \to \mathrm{Pin}^c(k) \to O(k) \to 1$ imply that δ induces an isomorphism on homotopy groups in dimensions less than or equal to $k-1$. Therefore δ induces an iomorphism

$$\tilde{\delta}^* \colon \tilde{H}^q(M\,\mathrm{Pin}^c(k+1); Z_2) \approx \tilde{H}^q(M\,\mathrm{Spin}^c(k) \wedge MO(1); Z_2) \quad \text{for } q < 2k-2.$$

We conclude that

$$\tilde{\delta}_* \colon \pi_{n+k+1}(M\,\mathrm{Spin}^c(k) \wedge MO(1)) \to \pi_{n+k+1}(M\,\mathrm{Pin}^c(k+1))$$

is an isomorphism on 2-primary components for $n < k - 4$. Since $\tilde{\delta} \colon M\,\mathrm{Spin}^c(k) \wedge MO(1) \to M\,\mathrm{Pin}^c(k+1)$ is compatible with the various inclusions, we can take the direct limit over k and obtain a 2-primary equivalence $\tilde{\Omega}_{n+1}^{\mathrm{Spin}^c}(MO(1)) \approx \Omega_n^{\mathrm{Pin}^c}$. We combine this calculation with Theorem 1 to show

THEOREM 2. *Let $A_{2k} = Z_{2^{k+1}}$ and let $A_{2k-1} = 0$. There is an isomorphism of graded Abelian groups*

$$\Omega_*^{\mathrm{Pin}^c} \approx Z[x_4, x_8, \ldots, x_{4j}, \ldots] \otimes A_* \oplus \tilde{H}_*(BZ_2; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})).$$

REMARK. The generators of A_{2k} can be taken to be real projective spaces RP^{2k} and the X_{4j} can be taken to be complex projective spaces CP^{2j}; we refer to Bahri and Gilkey [3] for details.

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