PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

MINIMAL SURFACES IN MANIFOLDS WITH S^1 ACTIONS AND THE SIMPLE LOOP CONJECTURE FOR SEIFERT FIBERED SPACES

JOEL HASS

ABSTRACT. The Simple Loop Conjecture for 3-manifolds states that if a 2-sided map from a surface to a 3-manifold fails to inject on the fundamental group, then there is an essential simple loop in the kernel. This conjecture is solved in the case where the 3-manifold is Seifert fibered. The techniques are geometric and involve studying least area surfaces and circle actions on Seifert Fibered Spaces.

1. Introduction. The Simple Loop Conjecture for 3-manifolds states the following:

SIMPLE LOOP CONJECTURE. *Let $f: F \to M$ be a 2-sided map from a closed 2-dimensional surface F to a 3-manifold M and let $f_\#: \pi_1(F) \to \pi_1(M)$ be the induced map on the fundamental group. If $\ker(f_\#) \neq 1$, then there is an essential simple closed curve in $\ker(f_\#)$.*

The analogous result for maps between two 2-dimensional surfaces was recently solved by D. Gabai [**Ga**]. This paper will present a solution to the conjecture for the case where M is a Seifert Fibered Space. This solution is actually a special case of a more general result concerning minimal codimension-one submanifolds of manifolds which admit a one-parameter group of isometries.

Throughout this paper we will assume that maps and manifolds are smooth. An immersed surface is *minimal* if it has mean curvature zero. An immersed codimension-one submanifold of a manifold is said to be 2-*sided* if its normal bundle is trivial. We say that a submanifold is a *strictly stable* minimal submanifold if there are no nearby submanifolds of smaller area, where by nearby we mean arbitrarily close in the C^0 topology. Given a one-parameter group of isometries I_t, $t \in R$, of a manifold, we call a submanifold F *vertical* if it is everywhere tangent to the orbits, and *horizontal* if each point x in F has an open neighborhood U in F such that $I_t(U) \cap U = \varnothing$ for $|t|$ sufficiently small. Thus vertical submanifolds are invariant under the isometries I_t while horizontal submanifolds are pushed off themselves locally. The main theorem on minimal submanifolds is the following result:

THEOREM 1. *Let M be a Riemannian n-manifold admitting a one-parameter group of isometries and let F be a strictly stable 2-sided minimal codimension-one submanifold immersed in M. Then F is either horizontal or vertical.*

REMARKS. 1. No assumption is made about injectivity properties of $\pi_1(F)$.

Received by the editors July 29, 1985 and, in revised form, December 16, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57N12; Secondary 53C42.
Research supported by National Science Foundation Postdoctoral Fellowship DMS 84-14097.

2. The theorem is false for 1-sided submanifolds. For example, it is possible to construct embedded 1-sided nonorientable incompressible surfaces in certain Lens spaces (those with nontrivial Z_2-homology). One can minimize area in the isotopy class of such surfaces [**MY**, **HS**] to obtain strictly stable nonorientable surfaces in Lens spaces. A Lens space is a Seifert Fibered Space over the 2-sphere with two exceptional fibers. Since horizontal surfaces in such a Seifert fibration are automatically orientable, being branch covers of the 2-sphere, and since vertical surfaces are compressible tori or Klein bottles, it follows that the theorem fails in this context.

3. A similar result to Theorem 1 has been obtained independently by Gao [**G**].

Here is an outline of how the Simple Loop Conjecture for Seifert Fibered Spaces follows from Theorem 1: An existence theorem for minimal surfaces establishes that if a homotopy class of maps of a surface into a 3-manifold contains no essential simple loops with null-homotopic image, then one can find a least area surface in that homotopy class. In most cases the natural locally homogenous metric that exists on this Seifert Fibered Space is invariant under a one-parameter group of isometries, and a least area map is then shown to be either horizontal or vertical. One can then check that horizontal and vertical maps always inject into the fundamental group. Some complications, due to the fact that one must sometimes pass to a double cover of a Seifert Fibered Space to guarantee existence of a one-parameter group of isometries, are then dealt with to complete the proof.

2. Results on minimal surfaces. The following lemma, sometimes known as the maximal principle for minimal surfaces, describes the intersection of two minimal surfaces near a point of tangency. Its proof is essentially the Hopf maximal principle of partial differential equations, applied to the equation for the difference of two minimal surfaces near a point of tangency [**MY**, **FHS**].

LEMMA 1. *Let M be a Riemannian 3-manifold and let F_1, F_2 be closed minimal surfaces immersed in M. Suppose that F_1 and F_2 are tangent at a point P. Then either F_1 and F_2 coincide on an open neighborhood of P or there is a C^1 coordinate chart (x^1, x^2, x^3) about P in which F_1 is given by $x^3 = 0$ and F_2 is given by $x^3 = \mathrm{Real}[(x^1 + ix^2)^n]$ for some $n \geq 2$.*

We will use the following existence result.

LEMMA 2. *Let M be an irreducible Riemannian 3-manifold with convex (possibly empty) boundary, and let F be a closed surface. Let $\mathcal{F} = \{f\colon F \to M \mid f$ is smooth and homotopic to g, and $\ker(g_\#)$ contains no nontrivial simple closed loop$\}$. Let $\mathcal{I} = \inf\{\mathrm{Area}(f) \mid f \in \mathcal{F}\}$. Then there exists a map f_0 in $\mathcal{F}$ with $\mathrm{Area}(f_0) = \mathcal{I}$. Moreover any such map is an immersion.*

PROOF. The existence statement is due essentially to Schoen-Yau and Sacks-Uhlenbeck [**SY**, **SU**]. Results of Osserman and Gulliver show that the map f has immersed image [**O**, **Gu**]. Finally Gabai's result shows that the map is actually an immersion, since if it had false branch points, i.e. branch covered its image, then there would be a simple loop in $\ker(f_\#)$.

NOTE. It is only assumed that the relevant homotopy class injects on simple loops, not necessarily all loops.

We now proceed to the proof of the main result on minimal surfaces in dimension three, which we restate below.

THEOREM 1. *Let M be a Riemannian n-manifold admitting a one-parameter group of isometries and let F be a strictly stable 2-sided minimal codimension-one submanifold immersed in M. Then F is either horizontal or vertical.*

PROOF. Suppose that F is neither vertical nor horizontal. Since F is a 2-sided immersion, there is a local isometry J mapping a tubular neighborhood of F in its normal bundle into M, $J: F \times [-\varepsilon, \varepsilon] \to M$. Let $F' = I_s(F)$ for some s with s close to 0. Then since F is not vertical, we can assume F' has distinct image from F, as if $I_t(F) = F$ for all small t, then F would be invariant under the isometries I_t and would be vertical. But since F is not horizontal, some point $P \in F$ has a neighborhood U such that $U \cap I_s(U) \neq \emptyset$ for any s sufficiently small.

We now consider the pull backs of F and F' to the normal bundle N of F. F pulls back to the zero section, and if s is sufficiently small, F' is a graph over F in N. Moreover F' intersects the zero section in a neighborhood of P. Note that both F and F' are strictly stable minimal submanifolds in N, as a smaller nearby submanifold in N would correspond to one in M. Also note that F' is a graph over F which takes on both positive and negative values. Let F'_+ be the part of F' which is a positive graph over F, and F'_- be the part which is a negative graph. Then F'_+ and F'_- intersect along F. Let $p: F \times [-\varepsilon, \varepsilon] \to F$ be the projection and let $F_+ = p(F'_+)$, $F_- = p(F'_-)$. We cut and paste F and F' to get two new submanifolds G_1 and G_2 by letting $G_1 = F_+ \cup F'_-$ and $G_2 = F_- \cup F'_+$. Then $\text{Area}(G_1) + \text{Area}(G_2) = 2\,\text{Area}(F)$ and each of G_1, G_2, are arbitrarily close to F in the C^∞ norm, so that we must have $\text{Area}(G_1) = \text{Area}(G_2) = \text{Area}(F)$ by the assumption of strict stability. But the minimal submanifold G_1 is not smooth along the codimension-one set of points in $F \cap F'$. It is shown in [**HK**] that this situation cannot occur. We outline this argument here. Let q be a point in this intersection set. Since F', G_1, G_2 are each C^∞ close to F, in a neighborhood of q we can realize each of them as graphs over a small ball B about q in F. Via results of Serrin [**Se**], a small neighborhood of B can be foliated by minimal balls parallel to B. It follows by monotonicity estimates and the maximal principle that, for a smaller ball B' about P, small graphs over $\partial B'$ bound minimal graphs over B' which are area-minimizing in their homology class (rel ∂). Such least area graphs have singularities of codimension at least seven [**L**], so we can pick a small disk about a nonsmooth point and replace it by a least area disk, decreasing the total area and yielding a surface nearby to G_1 with less area. This contradicts the assumption that F is strictly stable.

3. The Simple Loop Conjecture for Seifert Fibered Spaces. In this section we apply Theorem 1 to prove the Simple Loop Conjecture for Seifert Fibered Spaces. We begin by recalling the geometric properties of such manifolds. They admit natural geometric structures modelled on one of six geometries: E^3, S^3, $S^2 \times R$, $H^2 \times R$, NIL, and $\text{PSL}(2, R)$. A description of these geometries can be found in [**S**]. If M is a Seifert Fibered Space with its natural geometric structure, then either M or a double cover $\tilde{M}$ of M admit an action of S^1 where S^1 acts as isometries preserving the fibration. The obstruction to the existence of an S^1 action on M itself is the possible existence of vertical Klein bottles. We now state and prove the Simple Loop Conjecture for Seifert Fibered Spaces.

THEOREM 2. *Let $f: F \to M$ be a map from a closed 2-dimensional surface F to a closed Seifert Fibered Space M and let $f_{\#}: \pi_1(F) \to \pi_1(M)$ be the induced map on the fundamental group. If $\ker(f_{\#}) \neq 1$, then there is a nontrivial simple closed curve in $\ker(f_{\#})$.*

PROOF. Equip M with its natural geometric structure to make it into a Riemannian manifold. Assume that $\ker(f_{\#})$ contains no simple loop. We will show that $f_{\#}$ is injective.

We first homotop f to be a least area map in its homotopy class, using Lemma 2. Then by Lemma 2, f is an immersion.

Case 1. There is an S^1 action on M.

Then $f(F)$ is vertical or horizontal by Theorem 1. If $f(F)$ is vertical, then its image is either a torus or a Klein bottle. If this torus or Klein bottle fails to inject into $\pi_1(M)$, then the image of $\pi_1(F)$ in $\pi_1(M)$ is cyclic, and thus there is a simple loop in $\ker(f_{\#})$, contradicting our assumption. It follows that the image of f is a vertical torus or Klein bottle which injects into $\pi_1(M)$. Then by the surface case of the simple loop conjecture, F must be a torus or Klein bottle which injects into $\pi_1(M)$.

If F is horizontal, consider its preimage in U, the universal cover of M. U is one of the six geometries mentioned previously, and the preimage of F in U has components which are each transverse to the fibering of U by lines which are the preimages of the Seifert fibers. In each of the geometries this fibering is diffeomorphic to either the fibering of R^3 by lines parallel to the Z-axis or to the fibering of $S^2 \times S^1$ by the S^1 factors. Since such transverse surfaces in U can only be planes or 2-spheres, they inject into $\pi_1(U)$, and it follows that $\pi_1(F)$ must inject into $\pi_1(M)$.

Case 2. There is no S^1 action on M.

There is still an S^1 action on $\tilde{M}$, a double cover of M. If F lifts to $\tilde{M}$, then the previous case applies to show that the lift of F injects into $\pi_1(\tilde{M})$ and thus F injects into $\pi_1(M)$. Thus we assume that F fails to lift to $\tilde{M}$. A double cover $\tilde{F}$ of F does lift to $\tilde{M}$.

Claim. $\tilde{F}$ is strictly stable in $\tilde{M}$.

PROOF. Suppose not. The involution τ of $\tilde{M}$ with quotient M acts on $\tilde{F}$ with quotient F. Since $\tilde{F}$ is immersed, there is an $\varepsilon > 0$ such that τ induces an involution on $\tilde{T}$, an ε-tubular neighborhood of $\tilde{F}$, with quotient T, an ε-tubular neighborhood of F. There is a local isometry from $\tilde{N}$, the ε-normal bundle of $\tilde{F}$, to $\tilde{T}$ and from N, the ε-normal bundle of F, to T. By conformally blowing up the metric near ∂N, we can arrange that N has a metric h which is convex on its boundary and which is locally isometric to T for points within $\varepsilon/2$ of F. Thus $h = \varphi g$, where g is the original induced metric on the normal bundle of F and φ is a smooth real valued function on $F \times [-\varepsilon, \varepsilon]$ with $\varphi \geq 1$, and with $\varphi | F \times [-\varepsilon/2, \varepsilon/2] = 1$. The metric h lifts to a metric $\tilde{h}$ on $\tilde{N}$, with corresponding convexity properties.

Since $\tilde{F}$ is assumed to be not strictly stable, there is a surface $\tilde{G}$ arbitrarily C^0-close to $\tilde{F}$ in $\tilde{N}$ with $\text{Area}(\tilde{G}) < \text{Area}(\tilde{F})$. In particular we can find such a surface within distance $\varepsilon/2$ of $\tilde{F}$ in $\tilde{N}$. Let $\tilde{H}$ be the least area surface in $\tilde{N}$ which is homotopic to $\tilde{F}$ in $\tilde{N}$, with respect to the metric $\tilde{h}$. $\tilde{H}$ exists by Lemma 2. $\tilde{H}$ and $\tau\tilde{H}$ are both least area surfaces in N and it follows that they are each embedded and that they are either disjoint or equal [**FHS**, §2]. They cannot be disjoint as this would imply that F lifts to $\tilde{M}$. Thus $\tilde{H}$ covers a surface H in N which is homotopic to F and with

$$\text{Area}(H) = \text{Area}(\tilde{H})/2 < \text{Area}(\tilde{F})/2 = \text{Area}(F),$$

where area is measured in the metric h. Now F is a least area surface in N with respect to the metric g pulled back from M. Since the metric h on N is obtained from g by conformally multiplying by a function pointwise larger than one everywhere, it follows that the area of a surface contained in N with respect to the metric h is greater than or equal to the area with respect to the metric g. Thus the area of H with respect to g is less than the area of F with respect to g, contradicting our assumption that F is strictly stable with respect to the metric g. It follows that F is strictly stable in M, proving the claim.

Since $\tilde{M}$ does admit an S^1 action, it follows from the previous case that $\tilde{F}$ is horizontal or vertical in $\tilde{M}$, and thus that the same is true for F in M. This concludes the proof of the Simple Loop Conjecture for Seifert Fibered Spaces.

4. Other cases of the Simple Loop Conjecture. We remark here that in attempting to solve the Simple Loop Conjecture for the case of a general 3-manifold, the following special cases can be demonstrated.

1. *If $f: F \to M$ lifts to a cover of M in which it is homotopic to an embedding, then the Simple Loop Conjecture holds for f.*

In this case the result follows by using the loop theorem on the manifold obtained by cutting open along the embedded surface homotopic to the lift of f.

2. *If the Simple Loop Conjecture holds for M irreducible, then it holds in general.*

Let $f: F \to M$ be a map into a reducible 3-manifold M which injects on simple loops. Let S be an essential 2-sphere in M. Making f transverse to the 2-sphere S, the preimage of S on F is a collection of embedded curves with inessential image in $\pi_1(M)$. Thus these curves are all inessential in F and bound disks on F. By surgering these disks, we obtain a new map $f': F \to M$ which misses all the essential 2-spheres and induces the same map on $\pi_1(F)$ as f.

3. *If the Simple Loop Conjecture holds for $f_{\#}: \pi_1(F) \to \pi_1(M)$ onto, then it holds in general.*

Let M_1 be the cover of M corresponding to the subgroup $f_{\#}(\pi_1(F))$ of $\pi_1(M)$. f lifts to a map $f_1: F \to M_1$ such that $f_{1\#}: \pi_1(F) \to \pi_1(M_1)$ is onto. $f_{\#}$ has a simple loop in its kernel if and only if $f_{1\#}$ does. Similarly $f_{\#}$ is injective if and only if $f_{1\#}$ is injective.

4. *The Simple Loop Conjecture holds for general* 3*-manifolds if F is a torus.*

The image of $\pi_1(F)$ in $\pi_1(M)$ is an abelian group. If it is cyclic, then there is an infinite cyclic subgroup of $\pi_1(F) = Z + Z$ which is in the kernel of $f_{\#}: \pi_1(F) \to \pi_1(M)$. A generator of the kernel then gives a simple loop in kernel $(f_{\#})$. If the image of $f_{\#}$ is isomorphic to $Z + Z$, then $f_{\#}$ is injective. Finally, if the image is isomorphic to $Z + Z_p$ for some p, then 3-manifold topology dictates that the image is isomorphic to $Z + Z_2$ and that in this case M is $S^1 \times RP^2$ connect sum some three balls and homotopy spheres. In this case the torus would be 1-sided if its image in $\pi_1(M)$ were $Z + Z_2$.

REFERENCES

[**FHS**] M. Freedman, J. Hass and G. P. Scott, *Least area incompressible surfaces in 3-manifolds*, Invent. Math. **71** (1983), 609–642.

[**G**] L. Z. Gao, *Applications of minimal surfaces theory to topology and Riemannian geometry constructions of negative Ricci curved manifolds*, Thesis, S.U.N.Y. at Stony Brook, 1985.

[**Ga**] D. Gabai, *The simple loop conjecture*, preprint.

[**Gu**] R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. **97** (1973), 275–305.

[**H**] J. Hass, *Minimal surfaces in Seifert fibered spaces*, Topology and Appl. **18** (1984), 145–151.

[**HK**] J. Hass and L. Karp, *Local properties of minimal submanifolds* (in preparation).

[**HS**] J. Hass and P. Scott, *Existence theorems for minimal surfaces in 3-manifolds*. I, preprint.

[**L**] H. B. Lawson, Jr., *Minimal varieties in real and complex geometry*, Univ. of Montreal Press, 1973.

[**MY**] W. Meeks and S. T. Yau, *Topology of three dimensional manifolds and the embedding problems on minimal surface theory*, Ann. of Math. **112** (1980), 441–484.

[**O**] R. Osserman, *A proof of the regularity everywhere of the classical solution to Plateau's problem*, Ann. of Math. **91** (1970), 550–569.

[**S**] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[**SU**] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981), 1–24.

[**SY**] R. Schoen and S. T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math. **110** (1970), 127–142.

[**Se**] J. Serrin, *The problem of Dirichlet for quasilinear elliptic equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A. **264** (1969), 413–496.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 1000 CENTENNIAL DRIVE, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Hebrew University, Givat Ram, Jerusalem, Israel