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A SUBSTITUTE OF L'HOSPITAL'S RULE FOR MATRICES

W. KRATZ

ABSTRACT. In this paper the following limit theorem is obtained: If A and B are (n,n)-matrices with $\operatorname{rank}(A^T, B^T) = n$, $A^T B = B^T A$, then $A(A+SB)^{-1}S \to 0$ as $S \to 0+$, i.e. $S \to 0$ where S is symmetric and positive definite. Some applications of this result are given to linear algebra (the behavior of $(A+\lambda B)^{-1}$ as $\lambda \to 0$) and to differential equations (the asymptotic behavior of Hamiltonian systems and of selfadjoint differential equations of even order).

1. Introduction. Let $A(x), B(x), C(x)$ be real (n,n)-matrices such that $A(x)$, $B(x), C(x)$ are continuous, $B(x), C(x)$ are symmetric, and $B(x)$ is nonnegative definite for $x \in \mathbf{R}$. We consider (n,n)-matrices $U(x), V(x)$ which solve the *Hamiltonian system* $U' = AU + BV$, $V' = CU - A^T V$ with $U^T(x)V(x) \equiv V^T(x)U(x)$ and $\operatorname{rank}(U^T(x), V^T(x)) \equiv n$ (so-called *conjoined basis* [**1**, Definition 1.2]). Moreover, we assume that the Hamiltonian system is *normal* [**1**, Lemma 5; **2**], i.e. the focal points of $U(x)$ (those x where $U(x)$ is singular) are isolated. Observe that if $U^{-1}(x)$ exists, then $Q(x) := V(x)U^{-1}(x)$ is a symmetric matrix-function solving the Riccati equation $Q' + A^T Q + QA + QBQ - C = 0$ [**1**, **2**, **3**]. These differential systems are the basis for variational analysis of the second variation or for Sturm-Liouville eigenvalue problems. The following identity (due to Picone) is an essential tool, e.g. for deriving comparison and oscillation results:

$$(d/dx)\eta^T Q\eta = \eta^T C\eta + \varsigma^T B\varsigma - (\varsigma - Q\eta)^T B(\varsigma - Q\eta)$$

whenever $U^{-1}(x)$ exists and whenever $\eta(x), \varsigma(x)$ belong to $C_1(\mathbf{R})$ with $\eta' = A\eta + B\varsigma$ [**1**, Theorem 7 with $\alpha = 0$; **2**, **5**]. If $U(x)$ is singular for some $x_0 \in \mathbf{R}$, then we need to know the behavior of $Q(x)$ as $x \to x_0$ when applying this identity. It is shown in [**1**] that

$$\lim_{x \to x_0} u^T(x)Q(x)u(x) = d^T U^T(x_0)V(x_0)d$$

for any vector-solution $u(x), v(x)$ of the Hamiltonian system with $u(x_0) = U(x_0)d$ for some $d \in \mathbf{R}^n$. In the scalar case, i.e. $n = 1$, this limit theorem follows from l'Hospital's rule, and in this sense the theorem and also our result below may be considered as a substitute of l'Hospital's rule for matrices.

In this paper we derive a limit theorem for matrices where no differential equation is involved and which generalizes the result above. More precisely, we prove that $A(A+SB)^{-1}S \to 0$ as $S \to 0+$, i.e., $S \to 0$ and S is symmetric and positive definite, whenever A and B are (n,n)-matrices with $\operatorname{rank}(A^T, B^T) = n$, $A^T B = B^T A$

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(Theorem 1 of §2). In §3 and §4 we give some applications of this theorem to linear algebra (we study the behavior of $(A+\lambda B)^{-1}$ as $\lambda \to 0$) and to differential equations (we derive the limit result above and also the asymptotic behavior of solutions of $L(y) = \lambda r y$ as $\lambda \to -\infty$, where $L(y)$ is a scalar, selfadjoint differential operator of even order). In §5 we discuss the assumptions on A, B, and S made in Theorem 1.

2. Main results. We use the following *notation.* If A and B are (n,n)-matrices, then $A \le B$ (resp., $A < B$) means that A and B are symmetric and $B - A$ is nonnegative definite (resp., positive definite). We denote by A^T the transpose of A, by I the identity matrix, and by $\|\cdot\|$ the Euclidean norm of a vector or the induced matrix-norm (i.e., $\max_{d\neq 0} \|Ad\|/\|d\|$) of a matrix. Moreover, we use a result which is included in [**1**, Proposition A 1(ii), (iii), Proposition 1, and formula (7′)].

PROPOSITION 1. *Suppose that A and B are (n,n)-matrices such that*

$$\text{rank}(A^T, B^T) = n, \qquad A^T B = B^T A. \tag{1}$$

Then the following assertions hold:

(i) $\text{kernel}(A^T B) = \text{kernel}(A) \oplus \text{kernel}(B)$, *where $\oplus$ denotes a direct sum;*

(ii) $A + SB$ *is regular for all S with $0 < S \le \varepsilon I$, if $\varepsilon > 0$ is sufficiently small;*

(iii) *there exist matrices A_1, B_1 such that the following identities hold:*

$$\begin{aligned} &A^T B = B^T A, \quad A_1^T B_1 = B_1^T A_1, \quad A A_1^T = A_1 A^T, \quad B B_1^T = B_1 B^T, \\ &A B_1^T - A_1 B^T = A^T B_1 - B^T A_1 = I. \end{aligned} \tag{2}$$

Moreover, A_1 and B_1 may be chosen to be regular.

REMARK. Matrices A_1, B_1 according to (iii) are constructed in the proof of [**1**, Proposition 1] by $A_1 = -(B+\varepsilon A)K^{-1}$, $B_1 = (A-\varepsilon B)K^{-1}$ with $K = A^T A + B^T B$ and with sufficiently small $\varepsilon > 0$.

The main result of this paper is given by

THEOREM 1. *If A and B are (n,n)-matrices with* (1), *then*

$$\lim_{S\to 0+} A(A+SB)^{-1} S = 0,$$

where $S \to 0+$ stands for $S \to 0$, S is a symmetric and positive definite (n,n)-matrix.

This theorem is a direct consequence of

LEMMA 1. *Given (n,n)-matrices A, B with* (1), *there exist $\varepsilon > 0$ and $c > 0$ such that*

$$0 \le A(A+SB)^{-1} S \le S + cS^2 \quad \textit{for all } S \textit{ with } 0 < S \le \varepsilon I.$$

PROOF. We assume throughout that $\varepsilon > 0$ is sufficiently small. Hence, $Q(S) := A(A+SB)^{-1} S$ exists for $0 < S \le \varepsilon I$ by Proposition 1(ii). Since

$$(A^T + B^T S) S^{-1} Q(S) S^{-1} (A + SB) = A^T S^{-1} A + B^T A$$

(use $S^T = S$) the matrix $Q(S)$ is symmetric and ≥ 0 for all $0 < S \le \varepsilon I$ (use that $c^T B^T A c = c_1^T B^T A c_1$, if $c = c_1 + c_2$ with $c_1 \in \text{image}(A^T), c_2 \in \text{kernel}(A)$ which is orthogonal to c_1, and use $B^TA = A^TB$). Now fix $\varepsilon > 0$ and S with $0 < S \leq \varepsilon I$, and let $S(x) := S + x(\varepsilon I - S)$. Then $S = S(0) \leq S(x) \leq S(1) = \varepsilon I$ for $0 \leq x \leq 1$. Since $Q(S) = S + S\{-B(A+SB)^{-1}\}S$ we obtain that $-B(A + S(x)B)^{-1}$ is symmetric. Moreover, if $0 \leq x \leq 1$, then

$$\frac{d}{dx}\{-B(A+S(x)B)^{-1}\} = \{B(A+S(x)B)^{-1}\}(\varepsilon I - S)\{B(A+S(x)B)^{-1}\} \geq 0.$$

This implies that

$$0 \leq Q(S) \leq S + S\{-B(A+S(1)B)^{-1}\}S \leq S + cS^2,$$

where $c > 0$ is chosen such that $-B(A+\varepsilon B)^{-1} \leq cI$. □

A first consequence of Theorem 1 is

COROLLARY 1. *Under the assumptions of Theorem* 1 *let* S_0 *be any symmetric* (n,n)*-matrix. Then*

$$\lim_{S \to S_0+} (A+S_0B)(A+SB)^{-1}(A+S_0B) = A + S_0B,$$

where $S \to S_0+$ *means* $S \to S_0$ *with* $S > S_0$.

PROOF. We have $A + SB = A + S_0B + (S - S_0)B$, $\operatorname{rank}(A^T + B^TS_0, B^T) = \operatorname{rank}(A^T, B^T) = n$, and $(A^T + B^TS_0)B = A^TB + B^TS_0B$ is symmetric by (1). Hence

$$\begin{aligned}&(A+S_0B)(A+SB)^{-1}(A+S_0B)\\ &\quad = A + S_0B - (A+S_0B)(A+S_0B+(S-S_0)B)^{-1}(S-S_0)B \to A + S_0B\end{aligned}$$

as $S \to S_0+$ by Theorem 1 with $A + S_0B$ instead of A. □

In applications to differential equations ([**1**, Theorem 4] or Propositions 4, 5 below) Theorem 1 is used in the following form.

COROLLARY 2. *Suppose that the assumptions of Theorem* 1 *hold and that* A_1, B_1 *are chosen such that* (2) *holds* (*Proposition* 1(iii)). *Then*

$$\lim_{S \to 0+} A(A+SB)^{-1}(A_1+SB_1)A^T = A_1A^T.$$

PROOF. We have $\lim_{S\to 0+} A(A+SB)^{-1}A = A$ by Corollary 1. Hence

$$\lim_{S \to 0+} A(A+SB)^{-1}(A_1+SB_1)A^T = \lim_{S \to 0+} A(A+SB)^{-1}AA_1^T = A_1A^T$$

since $A_1A^T = AA_1^T$ by (2). □

This corollary shows that for $c = A^Td \in \operatorname{image}(A^T)$,

$$\lim_{S \to 0+} c^T(A+SB)^{-1}(A_1+SB_1)c$$

exists and equals $d^TA_1A^Td$. Actually, these c's are the only ones for which the limit exists. This fact is stated in the following theorem. We omit the proof since it follows in the same way as Corollary 7 of [**1**] (using essentially the minimum-maximum principle, Proposition 1(i), and the fact that

$$P(S) := (A+SB)^{-1}(A_1+SB_1)$$

is strictly increasing, i.e., $P(S_1) < P(S_2)$ for $0 < S_1 < S_2 \leq \varepsilon I$).

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *hold and that* A_1, B_1 *are chosen such that* (2) *holds. Then*

$$\lim_{S\to 0+} c^T(A+SB)^{-1}(A_1+SB_1)c = -\infty \quad \textit{for all } c \notin \mathrm{image}(A^T).$$

REMARK. The assertions of Theorems 1 and 2 with $-\infty$ instead of $+\infty$ hold also for $S \to 0-$ (replace B by $-B$ and A_1 by $-A_1$). But the limits have to be *one-sided* as is shown in §5.

3. Applications to linear algebra. The result in this section is

PROPOSITION 2. *Let* A *and* B *be* (n,n)*-matrices with* (1). *Then*

$$Q = \lim_{\lambda\to 0} \lambda(A+\lambda B)^{-1}$$

exist, and the matrices Q *and* $P := BQ = \lim_{\lambda\to 0}\lambda B(A+\lambda B)^{-1}$ *are characterized by the properties*

(i) P *is symmetric, idempotent,* $\mathrm{rank}(P) = n - \mathrm{rank}(A)$, $PA = 0$;
(ii) $Q = (A^TA+B^TB)^{-1}B^TP$.

PROOF. By Lemma 1 (with $S = \lambda I$) there exist $\varepsilon > 0, c > 0$ such that $0 \le A(A+\lambda B)^{-1} \le I + \lambda c I$, and $0 \le A(A-\lambda B)^{-1} \le I + \lambda c I$ (with $-B$ instead of B) for all $0 < \lambda \le \varepsilon$. Hence, the matrix-elements of $A(A+\lambda B)^{-1}$ are bounded for $0 < |\lambda| \le \varepsilon$. Since these matrix-elements are rational functions of λ,

$$\lim_{\lambda\to 0} A(A+\lambda B)^{-1} = I - P$$

exists, and then $\lim_{\lambda\to 0}\lambda B(A+\lambda B)^{-1} = P$ with a symmetric matrix P. Since $K := A^TA + B^TB$ is regular by (1), it follows that

$$\lambda(A+\lambda B)^{-1} = K^{-1}\lambda\{A^TA(A+\lambda B)^{-1} + B^TB(A+\lambda B)^{-1}\} \to K^{-1}B^TP =: Q.$$

Then, of course, $P = BQ$, and (ii) is proved. Next, we have

$$QA = \lim_{\lambda\to 0}\lambda(A+\lambda B)^{-1}A = \lim_{\lambda\to 0}(\lambda I - \lambda^2(A+\lambda B)^{-1}B) = 0.$$

Hence $PA = 0$, and since P is symmetric, we obtain that $P(AA^T) = 0 = (AA^T)P$. Therefore P and AA^T are simultaneously orthogonally diagonalizable [**4**, Theorem 6.2], i.e. there exists an orthogonal matrix U such that

$$\text{(3)}\quad U^T(AA^T)U = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_1 = \mathrm{diag}(\lambda_1,\dots,\lambda_k), \quad \lambda_i > 0,\ k = \mathrm{rank}(A),$$

and

$$U^TPU = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad D_2 = \mathrm{diag}(\mu_{k+1},\dots,\mu_n).$$

Finally, if $AA^Tc = 0$, then

$$\begin{aligned} 0 &= (A^T+\lambda B^T)^{-1}A^Tc = A(A+\lambda B)^{-1}c \quad \text{(by symmetry)} \\ &= \lim_{\lambda\to 0} A(A+\lambda B)^{-1}c = c - Pc. \end{aligned}$$

Hence, every $c \in \mathrm{kernel}(AA^T)$, $c \neq 0$, is an eigenvector of P with eigenvalue 1. This implies that $\mu_{k+1} = \cdots = \mu_n = 1$, which completes the proof. □

REMARKS. Observe first that P is uniquely determined by property (i), more precisely: $P = U^T J U$, where U is an orthogonal matrix with (3) and where $J = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ with $(n-k, n-k)$-identity matrix I. Thus, the matrix P depends on A only (not on B). Observe moreover, that for every (n,n)-matrix A there exists a matrix B such that (1) holds. Finally, we note as a consequence of Proposition 2 that $QA = AQ = 0$ and $\operatorname{rank}(Q) = \operatorname{rank}(P) = n - \operatorname{rank}(A)$.

4. Applications to differential equations. First, we study the local behavior of matrix solutions of Hamiltonian systems resp. of Riccati equations as discussed in the introduction. We consider (n,n)-matrices $U(x), V(x)$ which solve the Hamiltonian system

$$U' = AU + BV, \qquad V' = CU - A^T V, \tag{4}$$

where we assume that

(5) $A(x)$, $B(x)$, $C(x)$ are continuous, $B(x), C(x)$ are symmetric, $B(x)$ is nonnegative definite for $x \in \mathbf{R}$, and where the differential system is normal [**1**].

Moreover, we assume that $U(x), V(x)$ is a *conjoined basis* of (4) [**1**, Definition 1, 2], i.e.,

$$\operatorname{rank}(U^T(x), V^T(x)) \equiv n, \qquad U^T(x)V(x) \equiv V^T(x)U(x) \tag{6}$$

holds.

Our first result is

PROPOSITION 3. *If $B(x_0)$ is regular (for some $x_0 \in \mathbf{R}$), then*

$$\lim_{x \to x_0} (x - x_0) U^{-1}(x) = Q$$

exists, and

$$Q = (U^T(x_0)B^{-1}(x_0)U(x_0) + V^T(x_0)B(x_0)V(x_0))^{-1} V^T(x_0)\sqrt{B(x_0)}P\sqrt{B^{-1}(x_0)},$$

where the matrix P is characterized by

(7) *P is symmetric, idempotent,* $\operatorname{rank}(P) = n - \operatorname{rank}(U(x_0))$, *and* $P\sqrt{B^{-1}(x_0)}U(x_0) = 0$.

In particular $QU(x_0) = 0$ and $\lim_{x \to x_0}(x - x_0)U^{-1}(x)U(x_0) = 0$ hold.

PROOF. By our assumptions $S := B(x_0) > 0$, and (4) implies

$$\begin{aligned} U(x) &= U(x_0) + (x - x_0)U'(x_0) + o(1)(x - x_0) \\ &= (I + o(1))(U(x_0) + (x - x_0)SV(x_0)) \quad \text{as } x \to x_0. \end{aligned}$$

Put $\tilde{A} := \sqrt{S^{-1}}U(x_0), \tilde{B} = \sqrt{S}V(x_0)$. Then $\tilde{A}, \tilde{B}$ satisfy (1), and

$$U^{-1}(x) = (\tilde{A} + (x - x_0)\tilde{B})^{-1}\sqrt{S^{-1}}(I + o(1)) \quad \text{for } x \to x_0.$$

Now, our assertions follow from Proposition 2. □

In case $B(x_0)$ is singular, we have the following result [**1**, Corollaries 6 and 7], which we mentioned in the introduction.

PROPOSITION 4. *If $u_i(x)$, $v_i(x)$ $(i=1,2)$ are vector-solutions of* (4), *then*

$$\lim_{x\to x_0+} u_1^T(x)V(x)U^{-1}(x)u_1(x) = +\infty$$

if $u_1(x_0) \notin \operatorname{image}(U(x_0))$ *and*

$$\lim_{x\to x_0} u_1^T(x)V(x)U^{-1}(x)u_2(x) = c_1^T V^T(x_0)U(x_0)c_2$$

if $u_i(x_0) = U(x_0)c_i \in \operatorname{image}(U(x_0))$ *for* $i = 1,2$.

PROOF. Let U_i, V_i $(i=1,2)$ solve (4) with the initial conditions $U_1(x_0) = V_2(x_0) = 0, V_1(x_0) = -U_2(x_0) = I$ [**1**, formula (8)], and put $S(x) = -U_2^{-1}(x)U_1(x)$ for $|x| \le \varepsilon$ ($\varepsilon > 0$ small enough). Then

$$S'(x) = U_2^{-1}(x)B(x)(U_2^{-1}(x))^T \ge 0$$

(note, $U_1^TV_2 - V_1^TU_2 \equiv I$ by the Wronskian identity [**1**, Lemma 2]), and therefore $S(x) \to 0$ as $x \to x_0+$ ($S(x) > 0$ for $x > x_0$ by normality). Moreover, let $\tilde{U}, \tilde{V}$ solve (4), where the initial conditions are chosen according to Proposition 1(iii), such that $\tilde{U}^T\tilde{V} \equiv \tilde{V}^T\tilde{U}, U^T\tilde{V} - V^T\tilde{U} \equiv I$ holds. Then

$$U(x) = U_1(x)V(x_0) - U_2(x)U(x_0), \qquad \tilde{U}(x) = U_1(x)\tilde{V}(x_0) - U_2(x)\tilde{U}(x_0),$$

and therefore

$$U^{-1}(x)\tilde{U}(x) = (U(x_0) + S(x)V(x_0))^{-1}(\tilde{U}(x_0) + S(x)\tilde{V}(x_0)).$$

Now, Corollary 2 and Theorem 2 imply that

$$\lim_{x\to x_0} U(x_0)U^{-1}(x)\tilde{U}(x)U^T(x_0) = \tilde{U}(x_0)U^T(x_0),$$

and

$$\lim_{x\to x_0+} c^TU^{-1}(x)\tilde{U}(x)c = -\infty$$

for all $c \notin \operatorname{image}(U^T(x_0))$. This result is contained in Theorem 4 and Corollary 7 of [**1**], and it implies our assertions as is shown essentially in [**1**, Corollary 6]. □

Finally we give an application to the asymptotic behavior (as $\lambda \to -\infty$) of solutions of the differential equation

$$L(y) = \lambda r y, \tag{8}$$

where $L(y)$ is a selfadjoint differential operator of even order, i.e.

$$L(y) = \sum_{\nu=0}^{n} (-1)^\nu (r_\nu y^{(\nu)})^{(\nu)},$$

and where we assume that

$$r_\nu \in C_\nu(\mathbf{R}), \quad \nu = 0,\dots,n, \ r \in C(\mathbf{R}), \text{ and } r > 0, \ r_n > 0 \text{ on } \mathbf{R} \tag{9}$$

holds. For any function $y \in C_{2n-1}(\mathbf{R})$ we define column-vectors $u(x) = (u_k(x))$, $v(x) = (v_k(x))$ by [**1**; **2**, p. 309]

(10)

$$u_k(x) = y^{(k)}(x), \quad v_k(x) = \sum_{\nu=k+1}^{n} (-1)^{\nu-k-1}(r_\nu y^{(\nu)})^{(\nu-k-1)} \quad \text{for } k = 0,\dots,n-1.$$

Then u, v solve a corresponding Hamiltonian system (4) with (5) iff y solves (8) [**1**, **2**]. Now, suppose that the functions $y_{ij}(x)$ $(i = 1, \ldots, n,\ j = 1, 2)$ with corresponding columns $u_{ij}(x), v_{ij}(x)$ are solutions of (8) with the initial conditions

$$\begin{aligned} &U_j(x_0) = (u_{1j}(x_0), \ldots, u_{nj}(x_0)) = A_j, \\ &V_j(x_0) = (v_{1j}(x_0), \ldots, v_{nj}(x_0)) = B_j \\ &\qquad \text{with } A_j^T B_j = B_j^T A_j,\ A_1^T B_2 - B_1^T A_2 = I \text{ for } j = 1, 2 \end{aligned} \tag{11}$$

(see Proposition 1(iii)). Then, of course, $U_j(x) = U_j(x; \lambda)$, and the following holds.

PROPOSITION 5. *If $x > x_0$, then*

$$\lim_{\lambda \to -\infty} A_1 U_1^{-1}(x; \lambda) U_2(x; \lambda) A_1^T = A_2 A_1^T,$$

and

$$\lim_{\lambda \to -\infty} c^T U_1^{-1}(x; \lambda) U_2(x, \lambda) c = -\infty \quad \textit{for all } c \notin \operatorname{image}(A_1^T).$$

PROOF. *Fix $x > x_0$*, and consider the solutions $\tilde{y}_{ij}(x)$ of (8) with the initial conditions $\tilde{U}_1(x_0) = \tilde{V}_2(x_0) = 0$, $\tilde{V}_1(x_0) = -\tilde{U}_2(x_0) = I$ (compare the proof of Proposition 4). Then $\lim_{\lambda \to -\infty} \tilde{U}_1^{-1}(x; \lambda) \tilde{U}_2(x; \lambda) = -\infty$, i.e.

$$S(\lambda) := -\tilde{U}_2(x; \lambda) \tilde{U}_1^{-1}(x; \lambda) \to 0+ \quad \text{as } \lambda \to -\infty,$$

by [**1**, Theorem 11]. The initial conditions imply $U_j(x) = \tilde{U}_1(x) B_j - \tilde{U}_2(x) A_j$ for $j = 1, 2$, and therefore

$$U_1^{-1}(x; \lambda) U_2(x; \lambda) = (A_1 + S(\lambda) B_1)^{-1} (A_2 + S(\lambda) B_2).$$

Now, our assertions follow from Corollary 2 and Theorem 2.

REMARK. Observe that U_j, V_j $(j = 1, 2)$ form a fundamental system of the Hamiltonian system (corresponding to (8)), and therefore every $u(x) = (y^{(k)}(x))$, where $y(x)$ solves (8), can be expressed in terms of U_1, U_2, i.e. $u(x) = U_1(x) c_1 + U_2(x) c_2$ for certain $c_1, c_2 \in \mathbf{R}^n$.

5. Examples and concluding remarks. In this section we discuss the assumptions of Theorem 1 and show that they are all essential for the limit to exist.

(i) The *assumptions* (1) on the matrices A and B are essential, since $(A + SB)^{-1}$, of course, does not exist if $\operatorname{rank}(A^T, B^T) < n$. Moreover, if

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S(x) = \begin{pmatrix} x^3/3 & -x^2/2 & 0 \\ -x^2/2 & x & 0 \\ 0 & 0 & x \end{pmatrix},$$

then $\operatorname{rank}(A^T, B^T) = 3$, $S(x) > 0$ for $x > 0$, but $A^T B \neq B^T A$. A simple calculation shows that $A + S(x) B$ is singular for all $x \in \mathbf{R}$.

(ii) The *limit* in Theorem 1 has to be *one-sided* in general. This follows from the example

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad S(x) = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}.$$

Then (1) holds, $S(x)$ is symmetric but *indefinite* for $x \neq 0$. Again, $A + S(x) B$ is singular for $x \in \mathbf{R}$.

(iii) The matrix S has to be *symmetric*, since if

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = I, \quad S(x) = \begin{pmatrix} x^3 & -x & 0 \\ x - x^3 & x & 0 \\ 0 & 0 & x \end{pmatrix},$$

then (1) holds, $c^T S(x)c > 0$ for all $c \in \mathbf{R}$, $c \neq 0$ and $0 < x < 2$, but again, $A + S(x)B$ is singular on $\mathbf{R}$.

Of course, in all examples, $S(x)$ may be changed 'slightly' so that $A + S(x)B$ is regular for $0 < x \leq \varepsilon$ ($\varepsilon > 0$ small), but the limit in Theorem 1 does not exist as $x \to 0+$.

Finally we give an example to show that no factor in $A(A + SB)^{-1}S$ can be dropped (as e.g. in the special case of §3). Put

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S(x) = \begin{pmatrix} x^7 & x^4/2 \\ x^4/2 & x \end{pmatrix}.$$

Then (1) holds and $S(x) > 0$ for $x > 0$. Hence, Theorem 1 applies, so that the matrices $A(A + S(x)B)^{-1}S(x), A(A + S(x)B)^{-1}A, S(x)B(A + S(x)B)^{-1}A$, $S(x)B(A+S(x)B)^{-1}S(x)$ tend to a limit as $x \to 0+$. But, in this example, the matrices $A(A+S(x)B)^{-1}, S(x)(A+S(x)B)^{-1}A$, and then also $S(x)B(A+S(x)B)^{-1}$, $S(x)(A + S(x)B)^{-1}S(x)$ do *not converge* as $x \to 0+$.

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UNIVERSITÄT ULM, ABTEILUNG MATHEMATIK V, D-7900 ULM, OBERER ESELSBERG, FEDERAL REPUBLIC OF GERMANY