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ON TORSION-FREE ABELIAN k-GROUPS

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ABSTRACT. It is shown that a knice subgroup with cardinality $\aleph_1$, of a torsion-free completely decomposable abelian group, is again completely decomposable. Any torsion-free abelian k-group of cardinality $\aleph_n$ has balanced projective dimension $\leq n$.

Introduction. Recently, Hill and Megibben introduced the concept of a knice subgroup in their study of abelian k-groups [**6**] and also while considering the local Warfield groups in [**5**]. In this paper, we introduce a modified definition of a knice subgroup of a torsion-free abelian group. This helps us to extend the results of Hill and Megibben [**6**] and also simplify the proofs of their main theorems. Specifically we show that a knice subgroup with cardinality $\leq \aleph_1$, of a torsion-free completely decomposable abelian group, is again completely decomposable. This enables us to prove that any torsion-free abelian k-group (in particular, a separable group) of cardinality $\leq \aleph_n$ has balanced projective dimension $\leq n$.

All the groups that we consider here are torsion-free and abelian. We generally follow the notation and terminology of L. Fuchs [**3**]. Let P denote the set of all primes. By a height sequence we mean a sequence $s = (s_p)$, $p \in P$, where each s_p is a nonnegative integer or the symbol ∞. If G is a torsion-free group and $x \in G$, then $|x|$ denotes the height sequence of x where, for each $p \in P$, $|x|_p$ denotes the height of x at the prime p. For any height sequence $s = (s_p)$, ps is the height sequence (t_p), where $t_p = s_p + 1$ and $t_q = s_q$ for all $q \neq p$. $G(s)$ denotes the subgroup $\{x \in G\colon |x| \geq s\}$. $G(s^*)$ is the subgroup generated by the set $\{x \in G(s)\colon \sum_{p\in P}(|x|_p - s_p)$ is unbounded$\}$. Two height sequences (s_p) and (t_p) are said to be equivalent if $\sum_{p\in P} |s_p - t_p|$ is finite.

Following Hill and Megibben [**6**], we write $G(s^*, p)$ for $G(s^*) + G(ps)$. If $a \in G$, then $\langle a\rangle_*$ denotes the pure subgroup generated by a.

DEFINITION [**6**]. Let G be a torsion-free group.

(i) A subgroup H of G is said to be *$*$-pure* if H is pure and, for all height sequences s and primes p, $H \cap G(s^*, p) = H(s^*, p)$.

(ii) An element x of G is said to be *primitive* if for all height sequences s equivalent to $|x|$ and for all primes p for which $s_p = |x|_p \neq \infty$, $x \notin G(s^*, p)$. As pointed out in [**6**], x is primitive if and only if nx is primitive for any nonzero integer n. A useful remark is that if x is primitive and $x \in G(s^*, p)$, then $x \in G(ps)$ or $\sum_{p\in P}(|x|_p - s_p)$ is unbounded.

As pointed out in [**6**], $*$-purity is transitive, inductive, and inherited by direct summands.

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PROPOSITION 1. *$y \in G$ is primitive if and only if the pure subgroup $\langle y\rangle_*$ generated by y is $*$-pure.*

PROOF. Let y be primitive. In view of the preceding remarks, every nonzero element of $\langle y\rangle_* = H$ is primitive. Thus if $a \in H \cap G(s^*, p)$, then by the primitivity of a, either $a \in H \cap G(ps) = H(ps)$ or $\sum(|a|_p - s_p)$ is unbounded. Hence $a \in H(ps)$ or $a \in H(s^*)$. Thus H is $*$-pure. Conversely, suppose $H = \langle y\rangle_*$ is $*$-pure, where we can choose $|y| = s =$ the type of H. Clearly $H(s^*) = 0$. Since $y \notin H(ps)$, $y \notin H(s^*, p) = H \cap G(s^*, p)$ for all primes p for which H is not p-divisible. So y must be primitive.

The following modification of a result of Hill and Megibben is crucial in later discussions.

PROPOSITION 2. *If G is separable, then any finite rank $*$-pure subgroup H is a completely decomposable summand.*

PROOF. The proof is by induction on the rank of H. Let H have rank one, say, $H = \langle y\rangle_*$. Since H is contained in a completely decomposable summand of G, we may assume $G = \langle x_1\rangle * \oplus \cdots \oplus \langle x_n\rangle *$ and $y = x_1 + \cdots + x_n$. Clearly $|y| \leq |x_i|$, $i = 1, \ldots, n$. Since, by Proposition 1, y is primitive, at least one of the x_i's must have the same type s as y (as $y \notin G(s^*)$). Rearranging the x_i, if necessary, write $y = y' + z$, where $y' = x_1 + \cdots + x_k$ with $x_1, \ldots, x_k$ having the same type as y and $z = x_{k+1} + \cdots + x_n$. Then $\langle y'\rangle_*$, being pure, is a summand of the homogeneous completely decomposable group $\langle x_1\rangle * \oplus \cdots \oplus \langle x_k\rangle *$ and hence a summand of G, say $G = \langle y'\rangle_* \oplus N$ with $z \in N$. Then $G = \langle y\rangle_* \oplus N$.

Suppose H has finite rank $n > 1$ and that the result holds for $*$-pure subgroups of smaller rank. Let $y \in H$ be an element of maximal type s. We claim that y is primitive. Since this happens if ny is primitive, we may assume without loss of generality that $|y| = s$. Then $H(s^*) = 0$ and for each prime p for which $s_p = |y|_p \neq \infty$, $y \notin H(ps) = H(s^*, p) = H \cap G(s^*, p)$. Thus y is primitive in G. Then we can write $G = \langle y\rangle * \oplus K$ and $h = \langle y\rangle * \oplus L$, where $L = H \cap K$ is $*$-pure of rank $n - 1$ and hence a completely decomposable summand of G contained in K. This proves the result.

As a corollary we obtain a simple 'natural' proof of the classical theorem of Fuchs on summands of separable groups.

COROLLARY 3. *If G is separable and $G = A \oplus B$, then A is separable.*

PROOF. Let $p_A\colon G \to A, p_B\colon G \to B$ be coordinate projections. Let X be a finite subset of A. Then $X \subseteq G_1$, a finite rank completely decomposable summand of G. Now there is a finite rank completely decomposable summand G_2 of G such that

$$G_1 \subseteq p_A(G_1) + p_B(G_1) \subseteq G_2.$$

Proceeding like this we get an increasing chain of subgroups

$$G_1 \subseteq p_A(G_1) + p_B(G_1) \subseteq G_2 \subseteq p_A(G_2) + p_B(G_2) \subseteq \cdots.$$

If

$$G^* = \bigcup_{n=1,2,\ldots} G_n, \quad A^* = \bigcup_{n=1,2,\ldots} p_A(G_n), \quad \text{and} \quad B^* = \bigcup_{n=1,2,\ldots} p_B(G_n),$$

then $G^* = A^* \oplus B^*$. Since G^* is $*$-pure completely decomposable, so is A^*. Let $X \subseteq D$, a finite rank summand of A^*. Then D is $*$-pure in G. By the separability of G and Proposition 2, D is a summand of G. This proves that A is separable.

We now give a modified definition of a knice subgroup of a torsion-free group.

DEFINITION. (i) A subgroup N of a torsion-free group G is said to be *knice* if for each finite subset X of G, there is a finite rank completely decomposable subgroup H such that $N + H = N \oplus H$ is $*$-pure and $X \subseteq N \oplus H$.

(ii) A group G is called a *k-group*, if $\{0\}$ is knice.

It is clear that a knice subgroup is always balanced. Conversely, as pointed out in [**6**], a balanced subgroup B is knice in G if and only if G/B is a k-group.

REMARK. Our definition of knice is slightly different from that of Hill and Megibben [**5**, **6**]. For torsion-free groups it is readily seen that "knice" is our sense is the same as "pure knice" in the sence of Hill and Megibben [**6**].

We begin with a simple characterization of K-groups.

PROPOSITION 4. (i) *G is a k-group if and only if $G = C/B$, where C is completely decomposable and B is knice in C.*

(ii) *G is a k-group if and only if every countable subset of G can be embedded in a $*$-pure completely decomposable subgroup of G.*

PROOF. (i) Let G be a K-group. As with any torsion-free group we can write $G = C/B$, where C is completely decomposable and B balanced [**3**, p. 117, Exercise 16]. The rest follows from Theorem 4.3 in [**6**].

(ii) follows since the union of an increasing chain of finite rank $*$-pure completely decomposable subgroups is, by Proposition 2, again completely decomposable and $*$-pure.

COROLLARY 5. *A countable knice subgroup of a completely decomposable group is a summand.*

Using our modified definition of knice subgroups we give below a much simpler and direct proof of Theorem 4.8 of Hill and Megibben [**6**].

THEOREM 6 [**6**]. *If G is a K-group and H is knice in G, then H is a k-group.*

PROOF. Let X be a finite subset of H. Since G is a k-group, there is a finite rank $*$-pure completely decomposable subgroup A_1 of G containing X. Since H is knice, $A_1 \subseteq H \oplus M_1$, where M_1 is finite rank completely decomposable and $H \oplus M_1$ is $*$-pure. Actually $A_1 \subseteq H_1 \oplus M_1$, where $H_1 = f(A_1)$, f being the coordinate projection $H \oplus M_1 \to H$. Since $H_1 \oplus M_1$ has finite rank and G is a k-group, $H_1 \oplus M_1 \subseteq A_2$, a finite rank $*$-pure completely decomposable subgroup of G. Since H is knice,

$$A_2 \subseteq H_2 \oplus M_2 \subseteq H \oplus M_2,$$

where $H \oplus M_2$ is $*$-pure, $H_2 = f(A_2)$, $f\colon H \oplus M_2 \to H$ being the projection map and M_2 is finite rank completely decomposable. Here M_2 can be chosen to contain M_1: Indeed, if

$$L = H \oplus M_1 \subseteq H \oplus M_2,$$

then $L = H \oplus (L \cap M_2)$ by modular law. Since L is $*$-pure in G, so is $L \cap M_2$ and Proposition 2 implies that $L \cap M_2$ is a summand of M_2, say, $M_2 = (L \cap M_2) \oplus M_2'$. Then

$$H \oplus M_2 = H \oplus (L \cap M_2) \oplus M_2' = H \oplus M_1 \oplus M_2'.$$

Hence we can replace M_2 by $M_1 \oplus M_1'$ and assume that $M_1 \subseteq M_2$. Then $H_2 = f(A_2) \supseteq f(A_1) = H_1$. Proceeding like this, we get the following increasing chains of subgroups of G:

$$A_1 \subseteq H_1 \oplus M_1 \subseteq A_2 \subseteq H_2 \oplus M_2 \subseteq \cdots,$$
$$M_2 \subseteq M_2 \subseteq \cdots, \quad \text{and} \quad H_1 \subseteq H_2 \subseteq \cdots.$$

If

$$A^* = \bigcup_{n=1,2,\ldots} A_n, \quad M^* = \bigcup_{n=1,2,\ldots} M_n, \quad \text{and} \quad H^* = \bigcup_{n=1,2,\ldots} H_n,$$

then $A^* = M^* \oplus H^*$. Since, for each n, A_n is $*$-pure completely decomposable of finite rank, A_n is a summand of A_{n+1} by Proposition 2 and so A^* is $*$-pure completely decomposable. Hence its summand H^* is a $*$-pure completely decomposable subgroup of H containing X. Thus H is a k-group.

A modification of the above proof readily shows that a knice subgroup of a separable group is again separable. A natural question is about knice subgroups of completely decomposable groups. Should they necessarily be completely decomposable? Theorems 7 and 9 below consider this question.

THEOREM 7. *A knice subgroup B of a completely decomposable group C is $\aleph_1$-separable.*

PROOF. Let $A = C/B$. By Proposition 4(i), A is a k-group. Let $Y = \{y_1, y_2, \ldots\}$ be a countable subset of B. Write $C = \bigoplus_{i\in I} X_i$, where the X_i are rank one groups. Let $y_1 \in C_1$ a direct sum of finitely many X_i's. Since C/B is a k-group, $(C_1 + B)/B \subset A_1/B$, a $*$-pure finite rank completely decomposable subgroup of C/B. Since B is balanced, $A_1 = B \oplus S_1$, $S_1 \cong A_1/B$. Let $p\colon B \oplus S_1 \to B$ be the coordinate projection. Then $C_1 \subseteq B_1 \oplus S_1$, where $B_1 = p(C_1)$. Let C_2 be a direct sum of finitely many X_i's such that $\{y_2\} \cup (B_1 \oplus S_1) \subseteq C_2$. Again since C/B is a k-group, $(C_2 + B)/B \subseteq A_2/B$, a $*$-pure completely decomposable finite rank subgroup of C/B. By Proposition 2, $A_2/B = (A_1/B) \oplus \overline{S}$. Since, in addition, B is balanced, we can write $A_2 = B \oplus S_2$ with $S_2 \cong A_2/B$ and $S_1 \subseteq S_2$. Proceeding like this we get the following increasing chains:

$$C_1 \subseteq B_1 \oplus S_1 \subseteq C_2 \subseteq B_2 \oplus S_2 \subseteq \cdots,$$
$$S_1 \subseteq S_2 \subseteq \cdots, \quad \text{and} \quad B_1 \subseteq B_2 \subseteq \cdots.$$

If

$$C^* = \bigcup_{n-1,2,\ldots} C_n, \quad B^* = \bigcup_{n=1,2,\ldots} B_n, \quad \text{and} \quad S^* = \bigcup_{n=1,2,\ldots} S_n,$$

then $C^* = B^* \oplus S^*$. Since C^* is a summand of C, B^* is a completely decomposable summand (of C and) of B containing Y. Hence B is $\aleph_1$-separable.

PROPOSITION 8. *Each k-group G of cardinality $m > \aleph_0$ is the union of a smooth chain of k-groups $\{K_\alpha\}$, where each K_α has the cardinality less than m and is $*$-pure in G.*

PROOF. Let $\{x_\alpha\colon \alpha < \tau\}$ be a well-ordering of the elements of G. We construct the K_α inductively. Let $\beta < \alpha$. Suppose K_α have been constructed for all $\alpha < \beta$, so that $x_\gamma \in K_{\gamma+1}$ for all $\gamma < \alpha$, each K_α is a k-group, and is $*$-pure in G. If β is a limit ordinal, define $K_\beta = \bigcup_{\alpha<\beta} K_\alpha$. Suppose $\beta = \alpha + 1$. Let $K_{\alpha,1} = \langle x_\alpha \rangle + \sum S$, where S runs over all the $*$-pure finite rank completely decomposable subgroups of K. Let $K_{\alpha,2} = \sum S$, where S runs over all the $*$-pure finite rank completely decomposable subgroups of G containing finite subsets of $K_{\alpha,1}$, and so on. Define $K_\beta = \bigcup_{n=1,2,\dots} K_{\alpha,n}$. Clearly K_β is a k-group of cardinality $< m$. It is readily seen that K_β is $*$-pure in G.

Ou next theorem considers the knice subgroups of completely decomposable groups and extends Theorem 5.4 of [**6**].

THEOREM 9. *If G is completely decomposable, then any knice subgroup H of cardinality $\leq \aleph_1$, is completely decomposable.*

We first wish to review the notion of compatibility introduced by Hill [**4**] and a terminology connected with Hill's third axiom of countability.

Two subgroups A and B of G are said to be *compatible*, in symbols, $A\|B$, if the following condition holds: If $(a,b) \in A \times B$ and if s is a height sequence with $s \leq |a+b|$, then there exists an element $b' \in A \cap B$ such that $s \leq |a+b'|$. The proof of Lemma 1 of Hill [**4**] implies that if H is a balanced subgroup of G and if S is a subset of G, then there is a subgroup B of G such that $S \subset B$, $B\|H$, and $|B| = |S|$ if S is infinite. We shall also use the following remark pointed out in [**6**]. (in the proof of Theorem 5.4): If H is balanced in G and A is a pure subgroup of G with $A\|H$, then $A \cap H$ is a balanced subgroup of A. In order to prove Theorem 9, we need the following important lemma:

LEMMA 10. *Suppose G is completely decomposable of regular cardinality κ, H is a knice subgroup of G, $G = \bigcup_{\alpha<\kappa} G_\alpha$, and $G/H = \bigcup_{\alpha<\kappa} D_\alpha$ are smooth filtrations, where G_α are summands of G with $|G_\alpha| < \kappa$ and D_α are k-groups of $|D_\alpha| < \kappa$. Then there is a closed and unbounded subset C of κ such that $H = \bigcup_{\alpha\in C} H_\alpha$, where $H_\alpha = H \cap G_\alpha$ is knice in H for each $\alpha \in C$, and $(G_\alpha + H)/H = D_\alpha$.*

PROOF. Now $\bigcup_{\alpha<\kappa} D_\alpha = G/H = \bigcup_{\alpha<\kappa} (G_\alpha + H)/H$ are two κ-filtrations of G/H, so by a standard argument (cf. [**2**, p. 26]), $C = \{\alpha < \kappa | D_\alpha = (G_\alpha + H)/H\}$ is a closed and unbounded set. Let $H_\alpha = H \cap G_\alpha$ for all $\alpha \in C$. We claim that $D = \{\alpha \in C | G_\alpha \| H\}$ is also closed and unbounded. Since the union of an increasing chain of subgroups S each compatible to H is again compatible to H, D is closed. We wish to show that D is unbounded in C. Let $\alpha \in C$ so that $(G_\alpha + H)/H = D_\alpha$. Since H is balanced, by Lemma 1 of Hill [**4**], $G_\alpha \subseteq A_1$, where $A_1 \| H$ and $|A_1| = |G_\alpha|$. Since $\{G_\alpha\}$ is a κ-filtration of G, there is $\alpha < \alpha_1 \in C$ such that $A_1 \subseteq G_{\alpha_1}$. By Lemma 1 of Hill [**4**], there exists $A_2 \| H$ with $G_{\alpha_1} \subseteq A_2$ and $|A_2| = |G_{\alpha_1}|$. Proceeding like this we get an increasing chain

$$G_\alpha \subseteq A_1 \subseteq G_{\alpha_1} \subseteq A_2 \subseteq G_{\alpha_2} \subseteq \cdots, \quad \text{where } \alpha_i \in C.$$

Then $\bigcup_{n<\omega} G_{\alpha_n} = \bigcup_{n<\omega} A_n = G_\beta$, where $\beta = \lim \alpha_n$, is compatible to H and $\beta \in C$, since C is closed. Hence $\beta \in D$ and clearly $\alpha \leq \beta$. Thus D is unbounded. Now for $\alpha \in D$, $G_\alpha / H_\alpha \cong D_\alpha$ is a k-group, H is balanced and $G_\alpha \| H$. This implies that H_α is balanced in G_α, G_α/H_α is a k-group, and so H_α is knice in G_α and hence in G by the transitivity of kniceness (Proposition 4.10(i), [**6**]). Since H is knice in G, H_α is a knice subgroup of H, by Proposition 4.10(iv) of [**6**].

PROOF OF THEOREM 9. Since G is completely decomposable we can assume $|G| \leq \aleph_1$. As in Lemma 10, choose a smooth filtration $\{G_\alpha\}$ for G and $\{D_\alpha\}$ for G/H with stated properties. Proposition 8 guarantees the existence of the filtration

$\{D_\alpha\}$. As before let $H_\alpha = H \cap G_\alpha$. Since $|G/H| \le \aleph_1$, each D_α will be a countable k-group and hence completely decomposable. Now $H_{\alpha+1}/H_\alpha$ is countable, and H_α knice in $H_{\alpha+1}$ implies $H_{\alpha+1}/H_\alpha$ is a countable k-group and hence completely decomposable so that $H_{\alpha+1} = H_\alpha \oplus K_\alpha$. Then $H = \bigcup H_\alpha = \bigoplus K_\alpha$ is completely decomposable.

COROLLARY 11. *A k-group of cardinality $\aleph_1$ has balanced projective dimension ≤ 1.*

THEOREM 12. *Any k-group (in particular, a separable group) G of cardinality $\le \aleph_n$ has balanced projective dimension $\le n$.*

PROOF. By Corollary 11, the theorem holds if $n = 1$. Suppose the theorem is true if $1 \le n < m$. Let $|G| = \aleph_m$. By Proposition 4(i), $G = C/H$, where C completely decomposable and H a knice subgroup of C. By Lemma 8, $G = \bigcup_{\alpha<\aleph_n} D_\alpha$ is a smooth filtration of $*$-pure k-groups D_α with $|D_\alpha| \le \aleph_{m-1}$. By Lemma 10, this induces a smooth filtration $H = \bigcup_{\alpha \in C} H_\alpha$ of knice subgroups H_α of H, where C is a cub of $\aleph_m$. Then for each α, $H_{\alpha+1}/H_\alpha$ is a k-group of cardinality $\le \aleph_{m-1}$ and so has balanced projective dimension $\le m-1$.

Note that each H_α is in particular balanced in H. Then, by Auslander's Theorem [**1**], H has balanced projective dimension $\le m-1$. This implies that G has balanced projective dimension $\le m$.

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