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UNIQUENESS THEOREMS FOR SUBHARMONIC FUNCTIONS IN UNBOUNDED DOMAINS

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ABSTRACT. A theorem of Carlson says that a holomorphic function of exponential growth in the half-plane cannot approach zero exponentially along the boundary unless it vanishes identically. This paper presents a generalization of this result for subharmonic functions in a Greenian domain Ω, using the Martin boundary, minimal fine topology and PWB solution to the h-Dirichlet problem. Applications of the general theorem to specific choices of Ω, such as the half-space and strip, are presented in later sections.

1. Introduction. The closure and boundary of a subset F of Euclidean space $\mathbb{R}^n$ ($n \geqslant 2$) are denoted respectively by $\bar{F}$ and ∂F. By a hypoharmonic function defined on a domain Ω of $\mathbb{R}^n$ we mean a function which is either subharmonic or identically valued $-\infty$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way.

Let Γ denote the half-plane $\{z = x + iy \in \mathbb{C} : y > 0\}$. A theorem of Carlson says that a holomorphic function of exponential growth in Γ cannot approach zero exponentially along $\partial\Gamma$ unless it vanishes identically. More precisely, the following (cf. [**7**, 6.3.6]) can be deduced from Carleman's formula.

THEOREM A. *Let f be continuous in $\bar{\Gamma}$ and holomorphic in Γ. If*

(i) $$\liminf_{r \to \infty} r^{-1} \sup\{\log|f(z)| : z \in \Gamma, |z| = r\} < +\infty$$

and

(ii) $$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1 + x^2}\, dx < \int_{-\infty}^{\infty} \frac{\log^-|f(x)|}{1 + x^2}\, dx = +\infty,$$

then $f \equiv 0$ in Γ.

In §§2, 3 we state and prove a very general uniqueness theorem for hypoharmonic functions in a Greenian domain Ω. In general we replace the restrictions (i) and (ii) above by conditions involving harmonic measures. However, for particular choices of Ω, suitable estimates of harmonic measure are available, and specific Carlson-type theorems can be deduced. Several such examples, some new, are presented in §§4–7.

2. The general theorem. Let Ω be a domain in $\mathbb{R}^n$ with a Green kernel, and let Δ denote the Martin boundary of Ω, the compactified space $\Omega \cup \Delta$ being denoted by $\hat{\Omega}$. We use $\bar{F}^M$ and $\partial^M F$ to denote respectively the closure and boundary of a subset

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F of $\hat{\Omega}$ with respect to this topology. We will make use of the theory of the Perron-Wiener-Brelot solution to the h-Dirichlet problem for harmonic functions. For an account of this we refer the reader to [**6**, Chapter XVI] (or see [**8**, 1.VIII], where the problem is equivalently treated for relative harmonic functions).

Let h be a fixed positive harmonic function in Ω, and A be a nonempty closed subset of Δ with zero h-harmonic measure (this clearly rules out the possibility $\Omega = \mathbb{R}^n$). Now let (W_m) be a sequence of open subsets of $\hat{\Omega}$ such that $\overline{W}_m \subset W_{m+1}$ and $\bigcup_m W_m = \hat{\Omega} \setminus A$. We denote $W_m \cap \Omega$ by Ω_m, the set $\partial\Omega_m \cap \Omega$ by σ_m, and h-harmonic measure relative to Ω_m and $M \in \Omega_m$ by $\mu^h_{m,M}$. Our argument requires that Ω_1 have certain properties, so we insist that Ω_1 be a nontangentially accessible (NTA) domain. The precise definition of a NTA domain can be found in [**11**], but we give the relevant properties of such a domain in §3.1. We use ν to denote a fixed measure with compact support E satisfying $E \subset \overline{\Omega}_0 \subset \overline{\Omega}_1 \setminus \bar{\sigma}_1$ for some subdomain Ω_0 of Ω_1, and use h_* to denote a fixed positive harmonic function in Ω_1 which continuously vanishes on $\partial\Omega_1 \setminus \bar{\sigma}_1$.

Let $\mathscr{S}$ be the class of hypoharmonic functions s on Ω such that

$$(s/h)(Z) = \limsup_{X \to Z} s(X)/h(X) < +\infty \qquad (Z \in \Delta \setminus A).$$

Clearly, if $s \in \mathscr{S}$, then s/h is locally bounded above on $\hat{\Omega} \setminus A$. We define

$$\text{(1)} \qquad \lambda_m(s) = \int_E \lim_{X \to Z} \left\{ \int_{\sigma_m} s \, d\mu^1_{m,X}/h_*(X) \right\} d\nu(Z)$$

(for the existence of this expression see §3.1).

THEOREM 1. *If $s \in \mathscr{S}$, $(s/h)^+$ is h-resolutive for Ω, $(s/h)^-$ is not h-resolutive for Ω, and*

$$\text{(2)} \qquad \liminf_{m \to \infty} \lambda_m(s) < +\infty,$$

then $s \equiv -\infty$ in Ω.

3. Proof of Theorem 1.

3.1. We rely on the following properties of NTA domains.

THEOREM B (BOUNDARY HARNACK PRINCIPLE). *Let ω be a NTA domain, B be a relatively open subset of $\partial\omega$, and P_0 be a fixed point in ω. If ω_0 is a subdomain of ω such that $\partial\omega \cap \partial\omega_0 \subseteq B$, then there exists a constant c such that, for any positive harmonic functions h_1 and h_2 in Ω which vanish continuously on B and satisfy $h_1(P_0) = h_2(P_0)$, we have $h_1 \leqslant ch_2$ in ω_0.*

THEOREM C. *Let ω be a NTA domain and B be a relatively open subset of $\partial\omega$. If h_1 and h_2 are positive harmonic functions in ω which vanish continuously on B, then h_1/h_2 has a positive continuous extension to $\omega \cup B$.*

THEOREM D. *Let ω be a NTA domain and f be a resolutive function on $\partial\omega$. If f is continuous (in the extended sense) at a point Z of $\partial\omega$, then*

$$H^\omega_f(X) \to f(Z) \qquad (X \to Z),$$

where H^ω_f denotes the PWB solution of the 1-Dirichlet problem for f in ω.

Theorems B and C are given in [**11**, (5.1), (7.9)]. Theorem D, which asserts more than the regularity of ω, was originally proved for Lipschitz domains by Armitage [**1**, Theorem 2]. The proof given there is based on the Boundary Harnack Principle, and carries across to NTA domains.

It will now be shown that $\lambda_m(s)$, as defined in (1), exists and is real-valued if $s \in \mathcal{S}$ and $s \not\equiv -\infty$. (Clearly $\lambda_m(s) = -\infty$ if $s \equiv -\infty$.) Since s/h is bounded above on $\overline{\Omega}_m^M$, its restriction to $\partial^M\Omega_m$ is h-resolutive for Ω_m. In particular

$$h_m(X) = \int_{\sigma_m} s/h\, d\mu^h_{m,X} = \int_{\sigma_m} s\, d\mu^1_{m,X}$$

is harmonic in Ω_m. Further, since

$$h_m(X) = \int_{\sigma_1} h_m\, d\mu^1_{1,X} \qquad (X \in \Omega_1)$$

and Ω_1 is a NTA domain, it follows from Theorem D that each h_m continuously vanishes on $\partial\Omega_1 \setminus \bar{\sigma}_1$. By considering

$$\int_{\sigma_1} h_m^+\, d\mu^1_{1,X} \quad \text{and} \quad \int_{\sigma_1} h_m^-\, d\mu^1_{1,X},$$

it is now clear from Theorem C that h_m/h_* can be continuously defined on $\overline{\Omega}_1 \setminus \bar{\sigma}_1$. Hence $\lambda_m(s)$ exists as claimed.

3.2. We now proceed with the proof of Theorem 1 and suppose that $s \not\equiv -\infty$. Since $(s/h)^+$ is resolutive for Ω, the PWB solution of the h-Dirichlet problem for Ω with boundary data $(s/h)^+$ exists. We denote it by h_0.

We claim first that, for any $m \in \mathbb{N}$,

$$\text{(3)} \quad \text{mf} \limsup_{X \to Z,\ X \in \Omega_m} \{s(X) - h_0(X)\}/h(X) \leqslant 0 \quad \left(\text{a.e.}\ (\mu^h_{m,\cdot})Z \in \partial^M\Omega_m \cap \Delta\right),$$

where "mf" denotes that the upper limit is with respect to the minimal fine topology on $\hat{\Omega}$. According to [**13**, Théorème 25, Lemme 6], the points of $\partial^M\Omega_m \cap \Delta$ where $\Omega \setminus \Omega_m$ is not minimally thin form a set of zero $\mu^h_{m,\cdot}$-measure, and of the set of points of $\partial^M\Omega_m \cap \Delta$ where $\Omega \setminus \Omega_m$ is minimally thin, the subsets which have zero $\mu^h_{m,\cdot}$-measure coincide with those which have zero h-harmonic measure for Ω. Hence it is sufficient to establish that

$$\text{mf} \limsup_{X \to Z} \{s(X) - h_0(X)\}/h(X) \leqslant 0 \quad \left(\text{a.e.}\ (\mu^h)Z \in \Delta\right),$$

where μ^h denotes h-harmonic measure for Ω. Since

$$\text{mf} \lim_{X \to Z} h_0(X)/h(X) = (s/h)^+(Z) \quad \left(\text{a.e.}\ (\mu^h)Z \in \Delta\right),$$

it follows that

$$\text{(4)} \quad \text{mf} \limsup_{X \to Z} \{s(X) - h_0(X)\}/h(X) \leqslant \limsup_{X \to Z} s(X)/h(X) - (s/h)^+(Z)$$
$$= -(s/h)^-(Z) \leqslant 0 \quad \left(\text{a.e.}\ (\mu^h)Z \in \Delta\right)$$

as required.

Now let

$$f_m(X) = \begin{cases} [s(X) - h_0(X)]/h(X) & (X \in \sigma_m), \\ 0 & (X \in \partial^M \Omega_m \cap \Delta), \end{cases}$$

which is clearly $\mu^h_{m,\cdot}$-measurable. Since $s \in \mathscr{S}$, it follows that $(s - h_0)/h$ is bounded above on $\overline{\Omega}{}^M_m$, so a multiple of h belongs to the upper h-class for f_m in Ω_m. Further, from (3), the function $(s - h_0)$ belongs to the lower h-class. Hence f_m is h-resolutive for Ω_m, and we write the generalized solution as H_m. Since $s - h_0 \leqslant H_{m+1}$ in Ω_{m+1}, we have

$$H_m(M) \leqslant \int_{\sigma_m} H_{m+1}/h \, d\mu^h_{m,M} = H_{m+1}(M) \qquad (M \in \Omega_m),$$

whence (H_m) forms an increasing (wide sense) sequence of harmonic functions in Ω_m.

Next we claim that $\lim H_m = H$, say, is harmonic in Ω. To see this, consider the increasing sequence of nonnegative harmonic functions (H'_m) in Ω_2, where $H'_m = H_m - H_2$. Since

$$\int_{\sigma_1} H'_m \, d\mu^1_{1,M} = \int_{\sigma_1} H_m \, d\mu^1_{1,M} - \int_{\sigma_1} H_2 \, d\mu^1_{1,M} = H_m(M) - H_2(M) = H'_m(M),$$

it follows from Theorem D that H'_m vanishes continuously on $\partial\Omega_1 \setminus \bar{\sigma}_1$. Applying Theorem B to the functions h_* and H'_m in Ω_1 yields

$$H'_m(P_0) \leqslant c h_*(P_0) H'_m(X)/h_*(X)$$

whence

$$\begin{aligned} \nu(E) H'_m(P_0) &\leqslant c h_*(P_0)\{\lambda_m(s) - \lambda_m(h_0) - \lambda_2(s) + \lambda_2(h_0)\} \\ &\leqslant c h_*(P_0)\{\lambda_m(s) - \lambda_2(s) + \lambda_2(h_0)\}, \end{aligned}$$

and it follows from hypothesis (2) that $H(P_0)$ is finite. Thus H is a harmonic majorant of $s - h_0$ in Ω.

Now let m be arbitrary, and use the monotone convergence theorem to observe that

$$H(M) = \lim_{k \to \infty} H_k(M) = \lim_{k \to \infty} \int_{\sigma_m} H_k \, d\mu^1_{m,M} = \int_{\sigma_m} H \, d\mu^1_{m,M},$$

and so

$$\text{(5)} \qquad \operatorname{mf} \lim_{X \to Z} H(X)/h(X) = 0 \quad (\text{a.e. } (\mu^h) Z \in \Delta),$$

since $\bigcup W_m = \hat{\Omega} \setminus A$. Combining (4) and (5) we obtain

$$\operatorname{mf} \limsup_{X \to Z} \{s(X) - h_0(X) - H(X)\}/h(X) \leqslant -(s/h)^-(Z) \quad (\text{a.e. } (\mu^h) Z \in \Delta).$$

As $s - h_0 - H$ is nonpositive, it now follows that it is in the lower h-class for $-(s/h)^-$ in Ω. However $(s/h)^-$ is not h-resolutive (by hypothesis), so $s - h_0 - H \equiv -\infty$, contradicting our assumption that s is subharmonic.

4. Application to the half-space. A point of $\mathbb{R}^n$ will be denoted by $X = (X', x_n) = (x_1, \dots, x_n)$ and we write $|X| = (x_1^2 + \cdots + x_n^2)^{1/2}$. We use O to denote the origin. Surface area measure, where defined, will be represented by σ.

We will apply Theorem 1 to the half-space $\Omega = \mathbb{R}^{n-1} \times (0, +\infty)$ with $h \equiv 1$. Recall that the Martin compactification for Ω and the Alexandroff compactification for $\overline{\Omega}$ coincide. If (r_m) denotes an unbounded strictly increasing sequence of positive real numbers, then we can take W_m to be $\{X \in \overline{\Omega} : |X| < r_m\}$, and the set A will simply comprise the Alexandroff point (clearly A has 1-harmonic measure zero since the corresponding Martin kernel, x_n, is unbounded). Further, we take $E = \{O\}$ (so ν is a Dirac measure) and $h_*(X) = x_n$. The class $\mathcal{S}$ now consists of those hypoharmonic functions s on Ω such that

(6) $$s(Z) = \limsup s(X) < +\infty \qquad (X \to Z \in \partial\Omega).$$

Since it was shown in [**2**, Example 2] that

$$\lim_{X \to 0} (x_n)^{-1} \int_{\sigma_m} s \, d\mu^1_{m,X} = c r^{-n-1} \int_{\sigma_m} y_n s(Y) \, d\sigma(Y)$$

for a positive constant c depending only on n, and since a measurable function f on $\partial\Omega$ is 1-resolutive for Ω if and only if

$$\int_{\mathbb{R}^{n-1}} \frac{|f(X',0)|}{1 + |X'|^n} \, dX' < +\infty,$$

we can immediately write down the following uniqueness theorem. (A very similar result appears in [**12**, Theorem 2], but the method of proof is quite different.)

THEOREM 2. *If $s \in \mathcal{S}$, and*

(7)
$$\text{(i)} \quad \liminf_{r \to \infty} r^{-n-1} \int_{\{X \in \Omega : |X| = r\}} x_n s(X) \, d\sigma(X) < +\infty,$$

$$\text{(ii)} \quad \int_{\mathbb{R}^{n-1}} \frac{s^+(X',0)}{1 + |X'|^n} \, dX' < \int_{\mathbb{R}^{n-1}} \frac{s^-(X',0)}{1 + |X'|^n} \, dX' = +\infty,$$

then $s \equiv -\infty$ in $\overline{\Omega}$.

5. An application when $h \not\equiv 1$. Other choices of h in §4 were possible, and we give a simple alternative now. Let Ω, r_m, W_m, E, ν, and h_* be as in §4, and let (Z_m) be a sequence of points in $\partial\Omega$ with no finite limit point. Now let $\mathcal{S}$ be the class of hypoharmonic functions s on Ω such that

(8) $$\limsup_{X \to Z} s(X) \leqslant 0 \qquad (Z \in \partial\Omega \setminus \{Z_m : m \in \mathbb{N}\})$$

and

(9) $$a_m = \limsup_{X \to Z_m} (x_n)^{-1} |X - Z_m|^n s(X) < +\infty \qquad (m \in \mathbb{N}).$$

THEOREM 3. *If* $s \in \mathscr{S}$, (7) *holds, and*

$$\sum_{m=1}^{\infty} a_m^+/\{1+|Z_m'|^n\} < \sum_{m=1}^{\infty} a_m^-/\{1+|Z_m'|^n\} = +\infty, \tag{10}$$

then $s \equiv -\infty$ *in* $\bar{\Omega}$.

To see this, let (b_m) be a sequence of positive real numbers such that

$$\sum_{m=1}^{\infty} b_m/\{1+|Z_m'|^n\} < +\infty$$

and define

$$h(X) = (2x_n/c_n)\sum_{m=1}^{\infty} b_m|X-Z_m|^{-n} \qquad (X \in \Omega),$$

where c_n denotes the surface area of the unit sphere. Then h is a positive harmonic function in Ω and the Alexandroff point has zero h-harmonic measure. Further, it is not hard to see from (8) and (9) that

$$(s/h)(Z) = \limsup_{X \to Z} s(X)/h(X) < +\infty \qquad (Z \in \partial\Omega).$$

In fact, clearly

$$(s/h)(Z_m) = (c_n/2)(a_m/b_m) \qquad (m \in \mathbb{N})$$

whence (10) is equivalent to the h-resolutivity of $(s/h)^+$ and non-h-resolutivity of $(s/h)^-$ for Ω.

6. Application to the cylinder. In this and the following section we will confine our attention to the case $h \equiv 1$ for simplicity. Let $\Omega = \{X\colon |X'| < 1\}$ $(n \geqslant 2)$. We will employ the Bessel function $J_{(n-3)/2}$ defined in Watson [**14**, pp. 40–42]. The least positive zero of this function will be denoted by a_n and we write

$$\psi(t) = t^{(3-n)/2}J_{(n-3)/2}(a_n t) \qquad (t > 0).$$

Recall that the Martin compactification for Ω comprises $\partial\Omega$ and a point "at each end of" Ω (see [**9**, Theorem 8.12], for example). If (r_m) denotes an unbounded strictly increasing sequence of positive real numbers, then we can take W_m to be $\{X \in \bar{\Omega} : |x_n| < r_m\}$ and the set A will comprise the two adjoined points. Further, we take

$$E = \{X \in \bar{\Omega} : x_n = 0\},$$

$$d\nu(X) = \{\psi(|X'|)\}^2\, dX'\, d\delta_0(x_n) \qquad (X \in E),$$

and

$$h_*(X) = \psi(|X'|)\cosh(a_n x_n),$$

where δ_0 denotes the Dirac measure at the origin of $\mathbb{R}$. The class $\mathscr{S}$ now consists of those hypoharmonic functions s on Ω for which (6) holds.

THEOREM 4. *If* $s \in \mathcal{S}$ *and*

(i) $$\liminf_{r\to\infty} \operatorname{sech}(a_n r)\int_{\{X\in\Omega\,:\,|x_n|=r\}} \psi(|X'|)s(X)\,d\sigma(X) < +\infty,$$

(ii) $$\int_{\partial\Omega} e^{-a_n|x_n|}s^+(X)\,d\sigma(X) < \int_{\partial\Omega} e^{-a_n|x_n|}s^-(X)\,d\sigma(X) = +\infty,$$

then $s \equiv -\infty$ *in* $\overline{\Omega}$.

To see this, we refer to [**10**, §9] where it was shown that

$$\lambda_m(s) = \frac{1}{2}\operatorname{sech}(a_n r_m)\int_{\{X\in\Omega\,:\,|x_n|=r_m\}} \psi(|X'|)s(X)\,d\sigma(X).$$

As for condition (ii), it is easy to see that the function

$$v(X) = \psi(|X'|)e^{-a_n|x_n|}$$

is a potential in Ω which is harmonic in $\Omega\setminus E$. It can now be deduced from Theorem B that a nonnegative locally integrable function f on $\partial\Omega$ is resolutive if and only if

$$\int_{\partial\Omega} e^{-a_n|x_n|}f(X)\,d\sigma(X) < +\infty$$

as required.

Theorem 4 is a new result.

7. Application to the strip. Let $\Omega = \{X : 0 < x_n < 1\}$ and $\mathcal{S}$ be the class of hypoharmonic functions s on Ω for which (6) holds. The following is essentially [**3**, Theorem 2].

THEOREM 5. *If* $s \in \mathcal{S}$ *and*

(i) $$\liminf_{r\to\infty} r^{1-n/2}e^{-\pi r}\int_{\{X\in\Omega\,:\,|X'|=r\}} s(X)\sin(\pi x_n)\,d\sigma(X) < +\infty,$$

(ii) $$\int_{\mathbb{R}^{n-1}} (1+|X'|)^{1-n/2}e^{-\pi|X'|}\{s^+(X',0)+s^+(X',1)\}\,dX'$$
$$< \int_{\mathbb{R}^{n-1}} (1+|X'|)^{1-n/2}e^{-\pi|X'|}\{s^-(X',0)+s^-(X',1)\}\,dX' = +\infty,$$

then $s \equiv -\infty$ *in* $\overline{\Omega}$.

To see how this follows from Theorem 1, we need to recall that the Martin boundary for Ω comprises $\partial\Omega$ and a set, A, of points "at infinity" which can be identified with the boundary in $\mathbb{R}^{n-1}$ of the $(n-1)$-dimensional unit ball (see [**4**]). We take

$$W_m = \{X \in \overline{\Omega} : |X'| < r_m\}, \qquad E = \{X\in\overline{\Omega} : X' = O'\},$$
$$d\nu(X) = \sin^2(\pi x_n)\,d\delta_{O'}(X')\,dx_n \qquad (X\in E),$$

and

$$h_*(X) = |X'|^{(3-n)/2}I_{(n-3)/2}(\pi|X'|)\sin(\pi x_n),$$

where $\delta_{O'}$ is the Dirac measure at the origin O' of $\mathbb{R}^{n-1}$ and $I_{(n-3)/2}$ is the Bessel function defined in Watson [**14**, p. 77]. Condition (i) of the theorem is obtained by finding an explicit expression for $\lambda_m(s)$ in a manner directly analogous to that referred to in §6. The details of this are left to the reader. Condition (ii) can be written down immediately since necessary and sufficient conditions for the finiteness of the Poisson integral of s in Ω are given in [**5**, Lemma 1].

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