PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

ON ZERO-DIAGONAL OPERATORS AND TRACES

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ABSTRACT. A Hilbert space operator A is called zero-diagonal if there exists an orthonormal basis $\phi = \{e_j\}_{j=1}^{\infty}$ such that $\langle Ae_j, e_j\rangle = 0$ for all j. It is known that T is the norm limit of a sequence $\{A_k\}$ of zero-diagonal operators iff $0 \in W_e(T)$, the essential numerical range of T. Our first result says that if $0 \in W_e(T)$ and $\mathscr{J}$ is an ideal of compact operators strictly larger than the trace class, then the sequence $\{A_k\}$ can be chosen so that $|T - A_k|_{\mathscr{J}} \to 0$ ($\mathscr{J}$ cannot be replaced by the trace class!). If A is zero-diagonal, then the series $\sum_{j=1}^{\infty}\langle Ae_j, e_j\rangle$ converges to a value (zero) that can be called "the trace of A with respect to the basis ϕ". Our second result provides, for each operator T, the structure of the set of all possible "traces" of T (in the above sense). In particular, this set is always either the whole complex plane, a straight line, a singleton, or the empty set.

1. Introduction. In [3], the first author proved the following result for (bounded linear) operators acting on a complex separable infinite-dimensional Hilbert space $\mathscr{H}$: Let $T \in \mathscr{L}(\mathscr{H})$ (the algebra of all operators acting on $\mathscr{H}$); then the following are equivalent:

(i) There exists a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathscr{L}(\mathscr{H})$ such that $\langle A_n e_j^{(n)}, e_j^{(n)}\rangle = 0$ for all j (for a suitable orthonormal basis $\{e_j^{(n)}\}_{j=1}^{\infty}$ depending on n) and $\|T - A_n\| \to 0$ $(n \to \infty)$.

(ii) $0 \in W_e(T)$, the *essential numerical range* of T (see definition and properties in [4]).

This result can be improved to a "Weyl-von Neumann-Kuroda type" theorem: Let $\mathscr{K}$, $\mathscr{C}_1$ and $\mathscr{J}$ denote the ideal of all compact operators, the ideal of all trace class operators, and a normed ideal of compact operators with norm $|\cdot|_{\mathscr{J}}$ strictly weaker than the trace norm $|\cdot|_1$; then (i) (or (ii)) is also equivalent to:

(iii) There exists a sequence of *zero-diagonal operators* $\{A_n\}_{n=1}^{\infty}$ (i.e., operators satisfying the condition of (i) [3]), such that $T - A_n \in \mathscr{J}$ for all n, and $|T - A_n|_{\mathscr{J}} \to 0$ $(n \to \infty)$; furthermore, the result is false if $\mathscr{J}$ is replaced by $\mathscr{C}_1$.

The fact that $\mathscr{J}$ cannot be replaced by $\mathscr{C}_1$ produces a lot of trouble in connection with the second problem considered here.

Following [2], we shall say that an orthonormal basis $\phi = \{e_j\}_{j=1}^{\infty}$ belongs to $\mathrm{Dom}\{\mathrm{tr}\, T\}$ if the series $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ is convergent. In this case, we denote the

Received by the editors August 5, 1985 and, in revised form, January 29, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A12, 47A55, 47B10.
Key words and phrases. Zero-diagonal, essential numerical range, trace, set of traces, trace class operators.
The research of the third author has been partially supported by a Grant of the National Science Foundation.

complex number $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ by $\operatorname{tr}_\phi T$ (= the "trace" of T with respect to the ONB ϕ). The image of $\operatorname{tr} T$ will be denoted by $R\{\operatorname{tr} T\}$. Of course, we can have $\operatorname{Dom}\{\operatorname{tr} T\} = \varnothing$, and consequently $R\{\operatorname{tr} T\} = \varnothing$.

In **[2]**, A. Ben-Artzi affirmatively answered a conjecture of I. C. Gohberg by showing that, if T is compact, then $R\{\operatorname{tr} T\}$ is either empty, or a point, or a straight line, or the whole complex plane. But the results of **[2]** do not completely clarify which of these four possibilities correspond to a particular operator. It will be shown here that exactly the same four possibilities occur in the general case, that is, for not necessarily compact operators; moreover, the method of the present article is more geometric and direct than the one offered in the above reference, and provides an immediate identification of $R\{\operatorname{tr} T\}$ in terms of the real parts of $e^{i\theta}T$ ($0 \leqslant \theta < 2\pi$).

The authors are deeply indebted to Professors L. A. Fialkow, R. McGovern, and H. N. Salas for several helpful discussions, and to the referee for several suggestions to improve the presentation of the results.

2. Limits of zero-diagonal operators.

THEOREM 1 [**3**, THEOREM 3]. *A necessary and sufficient condition that an operator* $T \in \mathscr{L}(\mathscr{H})$ *is the norm-limit of zero-diagonal operators is that* $0 \in W_e(T)$.

As observed in **[3]**, a positive hermitian operator H with $0 \in \sigma_e(H)$ (the essential spectrum of H; see, e.g., **[4]**) is the norm-limit of zero-diagonals. However, H itself cannot be zero-diagonal (unless $H = 0$); furthermore, since a zero-diagonal operator A satisfies $\langle Ae_j, e_j\rangle = 0$ (with respect to a suitable ONB $\{e_j\}_{j=1}^{\infty}$), it is not difficult to conclude that, unless $H \in \mathscr{C}_1$, $H + C$ cannot be zero-diagonal for any trace class operator C. This example illustrates the worst possibility. Indeed, we have the following

THEOREM 1′. *Let* $T \in \mathscr{L}(\mathscr{H})$; *then* $0 \in W_e(T)$ (*and therefore T is norm-limit of zero-diagonal*) *if and only if*, *given a normed ideal* $\mathscr{J}$ *of compact operators strictly larger than* $\mathscr{C}_1$, *and* $\varepsilon > 0$, *there exists* $K_\varepsilon \in \mathscr{J}$, *with* $|K_\varepsilon|_{\mathscr{J}} < \varepsilon$, *such that* $T - K_\varepsilon$ *is a zero-diagonal operator*.

It will be convenient to cite some results from **[4]** (and some immediate consequences of the same techniques):

PROPOSITION 2 [**4**, THEOREM (5.1)]. *The following statements are equivalent for T in* $\mathscr{L}(\mathscr{H})$:

(i) $0 \in W_e(T) := \bigcap\{W(T+K)^- : K \in \mathscr{K}\}$ (*where* $W(A)$ *denotes the numerical range of the operator* A);

(ii) $0 \in \bigcap\{W(T+F)^- :$ *F is a finite rank operator* (*or*, *more generally*, *F runs over some ideal of compact operators*)$\}$;

(iii) *There exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *of unit vectors such that* $x_n \to 0$ (*weakly*), *and* $\langle Tx_n, x_n\rangle \to 0$ ($n \to \infty$);

(iv) *There exists an infinite orthonormal system (ONS)* $\{e_n\}_{n=1}^{\infty}$ *such that* $\langle Te_n, e_n\rangle \to 0$ $(n \to \infty)$; *moreover, given a finite dimensional subspace* $\mathcal{M}$, *the* e_n*'s can be chosen orthogonal to* $\mathcal{M}$ *(for all* $n = 1, 2, \dots$);

(v) *There exists an infinite-rank projection* P *such that* $PTP \in \mathcal{K}$;

(vi) *Given* $\varepsilon > 0$, *there exists an infinite-rank projection* P *such that* $PTP \in \mathcal{C}_1$, *and* $|PTP|_1 < \varepsilon$.

For instance, it is easy to see that a hermitian compact operator can be written as a 2×2 operator matrix with an infinite trace class operator in the $(1,1)$-entry. Moreover, the same is true for any compact operator. (Apply the result for hermitian compact operators to the real part, and then apply the same result to the compression of the imaginary part to the $(1,1)$-corner.) By using this observation, we deduce that (v) $\Rightarrow$ (vi). (For other equivalent conditions, see [**1** and **10**].)

Proof of Theorem 1$'$. The "if" part is a trivial consequence of (iv) $\Rightarrow$ (i) in Proposition 2.

Assume that $0 \in W_e(T)$. By using Proposition 2(vi), we can find $C_1 \in \mathcal{C}_1$, with $|C_1|_1 < \varepsilon/2$, an infinite-rank projection P (we can directly assume that $\ker P$ is also infinite dimensional), and orthonormal bases $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of $\ker P$ and, respectively, $\operatorname{ran} P$, such that

$$T - C_1 = \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & t_2 & * & & * & \\ & * & \ddots & & & \\ \hline & & & 0 & & \\ & * & & & 0 & * \\ & & & & * & \ddots \end{array}\right)\begin{matrix} f_1 \\ f_2 \\ \vdots \\ g_1 \\ g_2 \\ \vdots \end{matrix}.$$

Let $\{m_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. By using the well-known fact that an $m \times m$ complex matrix can always be written as

$$A = \begin{pmatrix} a/m & & & \\ & a/m & * & \\ & * & \ddots & \\ & & & a/m \end{pmatrix} \qquad (a = \operatorname{trace} A)$$

(with respect to a suitable ONB of its underlying space; see, e.g., [**5**]), we can find an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that

$$\bigvee\{e_n\}_{n=1}^{m_1} = \bigvee\{f_1, g_1, g_2, \dots, g_{m_1-1}\},$$
$$\bigvee\{e_n\}_{n=m_1+1}^{m_2} = \bigvee\{f_2, g_{m_1}, g_{m_1+1}, \dots, g_{m_1+m_2-2}\},$$

and so forth, and the diagonal of $T - C_1$ with respect to this ONB is the sequence $\{t_1/m_1, \dots, t_1/m_1$ (m_1 times), $t_2/m_2, \dots, t_2/m_2$ (m_2 times), $\dots, t_k/m_k, \dots, t_k/m_k$ (m_k times), $\dots\}$.

Thus, if C_2 is the normal diagonal operator defined by $C_2 =$ "diagonal of $T - C_1$", then $A = T - (C_1 + C_2)$ is a zero-diagonal operator. On the other hand, it follows from [**8**, pp. 525–529] that, if $m_k \to \infty$ fast enough, then $C_2 \in \mathcal{J}$, and

$|C_2|_{\mathscr{J}} < \varepsilon/2$. Thus, $K_\varepsilon = C_1 + C_2 \in \mathscr{J}$, and

$$|K_\varepsilon|_{\mathscr{J}} \leqslant |C_1|_{\mathscr{J}} + |C_2|_{\mathscr{J}} \leqslant |C_1|_1 + |C_2|_{\mathscr{J}} < \varepsilon.$$

The proof of Theorem 1′ is now complete. □

As a consequence of this result, we obtain the following particular example of a "rather general" situation (see [7]).

COROLLARY 3. *Let Δ_0 denote the class of all zero-diagonal operators; then*

$$(\Delta_0)^- = \Delta_0 + \mathscr{K}$$

is a closed subset of $\mathscr{L}(\mathscr{H})$. Indeed, given T in $(\Delta_0)^-$ and $\varepsilon > 0$, it is possible to write $T = A_\varepsilon + K_\varepsilon$, where $A_\varepsilon \in \Delta_0$, $K_\varepsilon \in \mathscr{K}$, and $\|K_\varepsilon\| < \varepsilon$.

3. The set of "traces" of a Hilbert space operator. In what follows, $\mathbf{Re}\,T = \frac{1}{2}(T + T^*)$ and $\mathbf{Im}\,T = (T - T^*)/2i$ denote the real and, respectively, the imaginary part of the operator T.

THEOREM 4. *Let $T \in \mathscr{L}(\mathscr{H})$; $R\{\operatorname{tr} T\}$ is either empty, or a point, or a line, or the whole complex plane $\mathbf{C}$; more precisely*:

(i) *$R\{\operatorname{tr} T\} = \mathbf{C}$ iff the positive part, $\mathbf{Re}(e^{i\theta}T)_+$, of $\mathbf{Re}(e^{i\theta}T)$ is not a trace class operator for any θ, $0 \leqslant \theta < 2\pi$.*

(ii) *$R\{\operatorname{tr} T\}$ is a line iff $\mathbf{Re}(e^{i\theta}T) \in \mathscr{C}_1$, but $\mathbf{Im}(e^{i\theta}T)_+, \mathbf{Im}(e^{i\theta}T)_- \notin \mathscr{C}_1$ for some θ.*

(iii) *$R\{\operatorname{tr} T\}$ is a point iff $T \in \mathscr{C}_1$.*

(iv) *$R\{\operatorname{tr} T\} = \varnothing$ iff $\mathbf{Re}(e^{i\theta}T)_+ \in \mathscr{C}_1$, but $\mathbf{Re}(e^{i\theta}T)_- \notin \mathscr{C}_1$ for some θ.*

For T in $\mathscr{L}(\mathscr{H})$, we define

$$\Gamma(T) = \left\{ \sum_{j=1}^{m} \langle Te_j, e_j\rangle : \{e_j\}_{j=1}^{m} \text{ runs over all finite ONS's} \right\}.$$

It can be easily checked that $\Gamma(T)$ is completely included in one of the two half-planes determined by a certain line l_ω with slope equal to $-\tan\omega$ if and only if $\mathbf{Re}(e^{i\omega}T)_+ \in \mathscr{C}_1$. This proves two things. First of all, we see that the properties $\mathbf{Re}(e^{i\omega}T)_+ \in \mathscr{C}_1$, $\mathbf{Re}(e^{i\omega}T)_- \notin \mathscr{C}_1$ (for some ω) imply that $\operatorname{Dom}\{\operatorname{tr} T\} = \varnothing$ and $R\{\operatorname{tr} T\} = \varnothing$. Secondly, we infer that $R\{\operatorname{tr} T\} = \mathbf{C}$ is impossible unless $\mathbf{Re}(e^{i\theta}T)_+ \notin \mathscr{C}_1$ for all θ.

LEMMA 5. *If $\mathbf{Re}(e^{i\theta}T)_+ \notin \mathscr{C}_1$ $(0 \leqslant \theta < 2\pi)$, then $\Gamma(T) = \mathbf{C}$.*

PROOF. It follows from our previous observations that the convex hull, $\operatorname{co}\Gamma(T)$, of $\Gamma(T)$ is equal to the whole plane. Furthermore, if $\mathscr{M}$ is a finite dimensional subspace and $\Gamma_{\mathscr{M}}(T)$ is defined exactly as $\Gamma(T)$, with the extra condition that $\bigvee\{e_j\}_{j=1}^m \perp \mathscr{M}$, then we also have $\operatorname{co}\Gamma_{\mathscr{M}}(T) = \mathbf{C}$.

Let $\lambda \in \mathbf{C}$, and let $\alpha = \sum_{j=1}^m \langle Tf_j, f_j\rangle \in \Gamma(T)$ ($\{f_j\}_{j=1}^m$ an ONS). If $\alpha = \lambda$, we are done. If not, let $\mathscr{M} = \bigvee\{f_j\}_{j=1}^m$. Since $\operatorname{co}\Gamma_{\mathscr{M}}(T) = \mathbf{C}$, we can find a convex polygon Δ with vertices

$$\beta = \sum_{j=1}^{n} \langle Tg_j, g_j\rangle,\ \beta' = \sum_{j=1}^{n'} \langle Tg_j', g_j'\rangle, \ldots, \beta^{(s)} = \sum_{j=1}^{n^{(s)}} \langle Tg_j^{(s)}, g_j^{(s)}\rangle$$

in $\Gamma_{\mathscr{M}}(T)$, including the linear segment $[\lambda, \alpha]$ in its interior. (Here $\{g_j\}_{j=1}^{m}$, $\{g_j'\}_{j=1}^{n'}, \ldots, \{g_j^{(s)}\}_{j=1}^{n^{(s)}}$ are ONS's orthogonal to $\mathscr{M}$.)

Let $\mathscr{N} = \bigvee\{\{f_j\}_{j=1}^{m}, \{g_j\}_{j=1}^{n}, \{g_j'\}_{j=1}^{n'}, \ldots, \{g_j^{(s)}\}_{j=1}^{n^{(s)}}\}$. Since $\operatorname{co}\Gamma_{\mathscr{N}}(T) = \mathbf{C}$, we can find a polygon Δ' with vertices in $\Gamma_{\mathscr{N}}(T)$, including Δ in its interior.

The line determined by λ and α intersects the boundary of Δ at a point α' such that $\lambda \in [\alpha, \alpha']$. We can assume that the vertices of Δ have been ordered so that $\alpha' \in [\beta, \beta']$. The complement (in $\mathbf{C}$) of the line determined by β and β' consists of two open half-planes, one of which contains λ and α, and the other contains one of the vertices,

$$\gamma = \sum_{j=1}^{p} \langle Th_j, h_j \rangle \qquad \left(\{h_j\}_{j=1}^{p} \text{ is an ONS orthogonal to } \mathscr{N}\right),$$

of Δ'.

Our hypotheses of T imply that $0 \in W_e(T)$. (Indeed, $0 \notin W_e(T)$ iff for some ω, and some $\varepsilon > 0$, the spectral projection of $\mathbf{Re}(e^{i\omega}T)$ corresponding to the half-line $(-\varepsilon, \infty)$ is finite dimensional.) Therefore (use Proposition 2(iv)), there exists an ONS $\{e_k\}_{k=1}^{\infty}$ with e_k orthogonal to $\mathscr{N} \vee (\bigvee\{h_j\}_{j=1}^{p})$ such that $\sum_{k=1}^{\infty}|\langle Te_k, e_k\rangle| < \infty$. Clearly, we can add a few vectors from this ONS to the ONS's constructed above, in order to obtain four ONS's with the same characteristics, determining points arbitrarily close to α, β, β', and γ, with the additional property that the four new systems have exactly the same number of vectors. In other words, we can directly assume that $m = n = n' = p$. It is straightforward to check that either λ belongs to the (close) triangle of vertices α, β, and γ, or λ belongs to the triangle with vertices α, β', and γ. Without loss of generality, we can assume that $\lambda \in \operatorname{co}\{\alpha, \beta, \gamma\}$; then $\lambda = c_1\alpha + c_2\beta + c_3\gamma$, $c_1, c_2, c_3 \geqslant 0$, $c_1 + c_2 + c_3 = 1$. Let $\mathscr{R}_j = \bigvee\{f_j, g_j, h_j\}$, $j = 1, 2, \ldots, m$. By applying the Hausdorff-Toeplitz theorem (se [**6**, Problem 166]) to $P_{\mathscr{R}_j}T|\mathscr{R}_j$, we can find unit vectors $d_j \in \mathscr{R}_j$ such that

$$\langle Td_j, d_j\rangle = c_1\langle Tf_j, f_j\rangle + c_2\langle Tg_j, g_j\rangle + c_3\langle Th_j, h_j\rangle \qquad (j = 1, 2, \ldots, m).$$

It readily follows that $\{d_j\}_{j=1}^{m}$ is an ONS, and

$$\sum_{j=1}^{m} \langle Td_j, d_j\rangle = c\sum_{j=1}^{m} \langle Tf_j, f_j\rangle + c_2\sum_{j=1}^{m} \langle Tg_j, g_j\rangle + c_3\sum_{j=1}^{m} \langle Th_j, h_j\rangle$$
$$= c_1\alpha + c_2\beta + c_3\gamma = \lambda.$$

We conclude that $\Gamma(T) = \mathbf{C}$. □

COROLLARY 6. *If* $\mathbf{Re}(e^{i\theta}T)_+ \notin \mathscr{C}_1$ $(0 \leqslant \theta < 2\pi)$, *then* $\mathscr{H}$ *has an ONB* $\{e_j\}_{j=1}^{\infty}$ *such that the series* $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ *is convergent, and*

$$\sum_{j=1}^{m_{4k+s+1}} \langle Te_j, e_j\rangle = \frac{i^s}{k} \qquad (s = 0, 1, 2, 3;\ k = 0, 1, 2, \ldots)$$

for a certain increasing sequence $\{m_k\}_{k=1}^{\infty}$ *of natural numbers.*

PROOF. Let $\{h_k\}_{k=1}^{\infty}$ be an ONB of $\mathscr{H}$, and let $\lambda_1 = \langle Th_1, h_1\rangle$. By our previous result, there exists an ONS $\{h_1, e_2', e_3', \ldots, e_{m_1}'\}$ such that

$$\langle Th_1, h_1\rangle + \sum_{j=2}^{m_1} \langle Te_j', e_j'\rangle = 1.$$

(To see this, observe that the proof of Lemma 5 actually shows that $\Gamma_{\mathscr{M}}(T) = \mathbf{C}$ for each finite dimensional subspace $\mathscr{M}$.)

Let $\mathscr{M}_1 = \vee\{h_1, e_2', \ldots, e_{m_1}'\}$. Since trace $P_{\mathscr{M}_1}T \mid \mathscr{M}_1 = 1$, $\mathscr{M}_1$ has an ONB $\{e_j\}_{j=1}^{m_1}$ such that

$$\langle Te_j, e_j\rangle = \frac{1}{\dim \mathscr{M}_1} = \frac{1}{m_1} \quad \text{for all } j = 1, 2, \ldots, m_1 \ [\mathbf{5}].$$

Let $h_2' \perp \mathscr{M}_1$ be any unit vector such that $h_2 \in \mathscr{M}_1 \vee \{h_2'\}$, and let $\lambda_2 = \langle Th_2', h_2'\rangle$. By a formal repetition of the above argument, we can find an ONS $\{e_j\}_{j=1}^{m_2}$ $(m_2 \gg m_1)$ such that

$$\langle Te_j, e_j\rangle = \begin{cases} \dfrac{1}{m_1} & \text{for } j = 1, 2, \ldots, m_1, \text{ and} \\ \dfrac{i-1}{m_2 - m_1} & \text{for } j = m_1 + 1, m_1 + 2, \ldots, m_2, \end{cases}$$

so that

$$\sum_{j=1}^{m_1} \langle Te_j, e_j\rangle = 1, \qquad \sum_{j=1}^{m_2} \langle Te_j, e_j\rangle = i.$$

By an obvious inductive argument, we can define an ONS $\{e_j\}_{j=1}^{\infty}$ and a strictly increasing sequence $\{m_k\}_{k=1}^{\infty}$ so that

$$\sum_{j=1}^{m_{4k+s+1}} \langle Te_j, e_j\rangle = \frac{i^s}{k} \qquad (s = 0, 1, 2, 3;\ k = 0, 1, 2, \ldots);$$

moreover, the m_k's can be chosen so that $m_{k+1} > 2m_k$ $(k = 1, 2, \ldots)$, and this is sufficient to guarantee that the series $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ converges. Clearly, the sum of such a series is equal to $\lim_{k\to\infty} 1/k = 0$.

Finally, observe that the above construction guarantees that $h_k \in \vee\{e_j\}_{j=1}^{m_k}$ for all $k = 1, 2, \ldots$, and this implies that the ONS $\{e_j\}_{j=1}^{\infty}$ is actually an ONB of $\mathscr{H}$. □

PROOF OF THEOREM 4. (i) We have already observed that $R\{\operatorname{tr} T\} = \mathbf{C} \Rightarrow \mathbf{Re}(e^{i\theta}T)_+ \notin \mathscr{C}_1$ for all θ.

Conversely, if $\mathbf{Re}(e^{i\theta}T)_+ \notin \mathscr{C}_1$ $(0 \leqslant \theta < 2\pi)$, then we can find an ONB $\{e_j\}_{j=1}^{\infty}$ satisfying the conditions of Corollary 6. It is a straightforward exercise to check that, given $\lambda \in \mathbf{C}$, the series $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ can be reordered as a new series $\sum_{j=1}^{\infty}\langle Te_{\pi(j)}, e_{\pi(j)}\rangle$ (where π is a reordering of the natural numbers depending on λ) convergent to λ. Therefore $R\{\operatorname{tr} T\} = \mathbf{C}$.

(ii) If $R\{\operatorname{tr} T\}$ is a line, then $T \notin \mathscr{C}_1$. It follows from (i) and our previous observations that $\mathbf{Re}(e^{i\omega}T) \in \mathscr{C}_1$ for some ω $(0 \leqslant \omega < 2\pi)$. Since $T \notin \mathscr{C}_1$, we deduce that $\mathbf{Im}(e^{i\omega}T)_+$ and $\mathbf{Im}(e^{i\omega}T)_-$ cannot be trace class operators.

On the other hand, if $T = e^{-i\omega}(A + iB)$, where A, B are hermitian, and $A \in \mathscr{C}_1$, but $B_+, B_- \notin \mathscr{C}_1$, then by using the same kinds of arguments as in the proofs of Lemma 5 and Corollary 6 we can show that $R\{\operatorname{tr} T\}$ is a line with slope equal to $\cot \omega$.

(iii) If T is a trace class operator, then $\sum_{j=1}^{\infty}\langle Te_j, e_j\rangle$ is convergent and equal to trace(T) for all possible ONB's of $\mathscr{H}$ [**9**]. Therefore, $R\{\operatorname{tr} T\}$ = one point.

If $R\{\operatorname{tr} T\}$ = one point, then it follows from (i), (ii), and our previous observations that $\mathbf{Re}(e^{i\theta}T)_+ \in \mathscr{C}_1$ for all θ $(0 \leqslant \theta < 2\pi)$, whence we conclude that $T \in \mathscr{C}_1$.

(iv) We have already observed that $R\{\operatorname{tr} T\} = \varnothing$ whenever $\mathbf{Re}(e^{i\omega}T)_+ \in \mathscr{C}_1$, but $\mathbf{Re}(e^{i\omega}T)_- \notin \mathscr{C}_1$ for some ω $(0 \leqslant \omega < 2\pi)$. On the other hand, it follows from (i), (ii), and (iii), that if $R\{\operatorname{tr} T\}$ is not a point, neither a line, or a plane, then $\mathbf{Re}(e^{i\omega}T)_+ \in \mathscr{C}_1$, but $\mathbf{Re}(e^{i\omega}T)_- \notin \mathscr{C}_1$ for some ω $(0 \leqslant \omega < 2\pi)$.

The proof of Theorem 4 is now complete. □

COROLLARY 7. *Let* $T \in \mathscr{L}(\mathscr{H})$.

(i) *If* $\operatorname{Dom}\{\operatorname{tr} T\} \neq \varnothing$, *then there exists an ONB* $\{e_j\}_{j=1}^{\infty}$ *such that*

$$R\{\operatorname{tr} T\} = \left\{ \sum_{j=1}^{\infty} \langle Te_{\pi(j)}, e_{\pi(j)} \rangle \right\},$$

where π *runs over all possible reorderings of the natural numbers that make the reordered series convergent.*

(ii) *Either* $T \in \mathscr{C}_1$ *and* $\sum_{j=1}^{\infty} \langle Te_j, e_j \rangle$ *is absolutely convergent (to trace* T*) for all possible ONB's* $\{e_j\}_{j=1}^{\infty}$ *of* $\mathscr{H}$, *or there exists an ONB that makes the series divergent.*

(iii) *If* $0 \in$ interior $W_e(T)$, *then* $R\{\operatorname{tr} T\} = \mathbf{C}$.

(iv) *If* $R\{\operatorname{tr} T\} \neq \varnothing$, *then* $0 \in W_e(T)$; *in this case* $R\{\operatorname{tr}(T + C)\} \neq \varnothing$ *for each* $C \in \mathscr{C}_1$.

(v) *If* 0 *is a boundary point of* $W_e(T)$, J *is a normed ideal of compact operators strictly larger than* $\mathscr{C}_1$, *and* $\varepsilon > 0$, *then there exists* $K_\varepsilon \in \mathscr{J}$, *with* $|K_\varepsilon|_{\mathscr{J}} < \varepsilon$, *such that* $R\{\operatorname{tr}(T + K_\varepsilon)\} = \mathbf{C}$.

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