PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

SOME CHARACTERIZATIONS OF TRIVIAL PARTS FOR $H^\infty(D)$

THOMAS J. ABRAM AND MAX L. WEISS

ABSTRACT. The unit disc in the complex plane is made into a locally compact topological group. This group acts as a transformation group on the maximal ideal space of the Banach algebra of bounded analytic functions on the disc. Among other characterizations the trivial parts are shown to be the minimal closed invariant sets of this transformation group. A point in the maximal ideal space is a trivial part if and only if it is the limit of a maximal invariant filter. An example shows that the correspondence between such points and filters is not one-to-one.

1. Introduction. We assume that the theory of bounded analytic functions as developed in Hoffman's book **[4]** is somewhat familiar to the reader. In addition we will rely heavily on the paper **[5]** characterizing the parts of the maximal ideal space of $H^\infty(D)$, where $D = \{z\colon |z| < 1\}$ is the unit disc in the complex plane and $T = \{z\colon |z| = 1\}$ is the unit circle. For an introduction to transformation groups we refer to **[3]**.

The present paper begins with the observation that the unit disc can be thought of as a group (see **[1]**). We let $z \in D$ correspond to the unique linear fractional transformation, L_z, which maps D onto D, sends 0 to z, and fixes the point 1 on T. These transformations form a group under composition which can also be considered as supplying D with a group structure. Explicitly we define $w \cdot z$ to correspond to $L_z \circ L_w$, for w, $z \in D$, so

$$w \cdot z = L_z(L_w(0)) = \frac{(1-z)w + z(1-\bar{z})}{\bar{z}(1-z)w + (1-\bar{z})}.$$

Then, the group identity is 0 and $z^{-1} = -z(1-\bar{z})/(1-z)$.

Supplying this group with the ordinary topology, the disc D becomes a locally compact topological group. The pseudo-hyperbolic metric

$$\rho(w,z) = \left|\frac{w-z}{1-\bar{z}w}\right| = |w \cdot z^{-1}|$$

is a right invariant metric giving the same topology to D.

In **[5]** Hoffman begins with the collection of maps $z \to (z+\alpha)/(1+\bar{\alpha}z)$, $\alpha \in D$, of D onto itself and extends these to be maps of D into the maximal ideal space, $\mathcal{M}$, of H^∞ in order to "compute" the Wermer analytic maps of D onto Gleason parts of $\mathcal{M}$. We borrow these results for the collection of self-maps $\{L_z\colon z \in D\}$ above. A

Received by the editors July 24, 1985 and, in revised form, December 23, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D50; Secondary 43A99.
Key words and phrases. Bounded analytic functions, parts, transformation group.

difference will be that since these maps form a group we will be able to use the terminology and tools of transformation groups.

Recall that if X is a topological space and G is a topological group with an action $\pi\colon G \times X \to X$, $\pi(g, x) = g \cdot x$ defined, then (G, X, π) is a (left) transformation group if $e \cdot x = x$ for all $x \in X$ (e, the group identity); $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$; π is continuous. Clearly D is a left transformation group over itself. Given $h \in \mathcal{M}$ and a net z_α in D, $z_\alpha \to h$, we define the action $z \cdot h = \lim z \cdot z_\alpha$. Adapting [5] to this situation one can easily obtain the next result.

THEOREM 1. *Let $\mathcal{M}$ be the maximal ideal space of the Banach algebra $H^\infty(D)$. With the group structure on D and the action of D on $\mathcal{M}$ defined above,*

(a) *D is a transformation group over $\mathcal{M}$.*

(b) *Given $h \in \mathcal{M}$ the map $\tau_h\colon D \to \mathcal{M}$ defined by $\tau_h(z) = z \cdot h$ is an analytic map (perhaps constant) of D onto the part, $P(h)$, of h.*

(c) *Thus, $f \circ \tau_h \in H^\infty(D)$ for each $f \in H^\infty(D)$ and $P(h)$ is the orbit, $\{z \cdot h\colon z \in D\}$, of h.*

PROOF. Theorem 4.3 of [5] shows that the action $z \cdot h$ is independent of the net tending to h; and, with $\tau_h(z) = z \cdot h$, that both τ_h and the transformation $h \to \tau_h$ are continuous. Let $(z_\alpha, h_\alpha) \to (z, h)$ in $D \times \mathcal{M}$, let $\varepsilon > 0$, and let $f \in H^\infty$. The family $\{f \circ \tau_{h_\alpha}\}$ is uniformly bounded and, according to the cited theorem, the functions $f \circ \tau_{h_\alpha}$ are analytic (perhaps, constant). The family is therefore uniformly equicontinuous on D. Thus, there is a $\delta > 0$ such that whenever $|w - z| < \delta$, $w \in D$, then $|(f \circ \tau_{h_\alpha})(w) - (f \circ \tau_{h_\alpha})(z)| < \varepsilon/2$. As we remarked above $\phi \to \tau_\phi$ is continuous. Thus, $\tau_{h_\alpha} \to \tau_h$, so $\tau_{h_\alpha}(z) \to \tau_h(z)$, and $f(\tau_{h_\alpha}(z)) \to f(\tau_h(z))$. Choose α so large that $|z_\alpha - z| < \delta$ and $|f(\tau_{h_\alpha}(z)) - f(\tau_h(z))| < \varepsilon/2$. Then, for such α,

$$\begin{aligned} |f(z_\alpha \cdot h_\alpha) - f(z \cdot h)| &= \left|f\left(\tau_{h_\alpha}(z_\alpha)\right) - f\left(\tau_h(z)\right)\right| \\ &\leqslant \left|f\left(\tau_{h_\alpha}(z_\alpha)\right) - f\left(\tau_{h_\alpha}(z)\right)\right| + \left|f\left(\tau_{h_\alpha}(z)\right) - f\left(\tau_h(z)\right)\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus, $f(z_\alpha \cdot h_\alpha) \to f(z \cdot h)$ for all $f \in H^\infty$. Consequently, if $(z_\alpha, h_\alpha) \to (z, h)$, then $z_\alpha \cdot h_\alpha \to z \cdot h$, showing that the map $(z, h) \to z \cdot h$ is continuous. It is now immediate that D is a left transformation group over $\mathcal{M}$. The remaining assertions are simple translations of the results of Theorem 4.3 to the present terminology.

One of the central concepts in the theory of transformation groups is that of *invariant sets* (S is invariant if $G \cdot S \subset S$) and in particular minimal closed invariant sets. It is routine to verify facts such as: any closed invariant set contains a minimal such; if S is invariant so is its closure; a subset S is invariant if and only if it is a union of orbits. Then, for example, since the orbits in $\mathcal{M}$ are exactly the parts, it is transparent that the closure of a part is a union of parts. It is equally transparent that trivial parts are minimal closed invariant sets. Later, after the converse has been proved, it will follow immediately that the closure of any part contains trivial parts.

The main results of this paper consist of the utilization of the elementary theory of transformation groups to throw new light on one point or trivial parts of $\mathcal{M}$.

2. Characterizations of trivial parts. Before proceeding to the statement of the main results we require additional facts about transformation groups as they apply to the present example.

Given $z \in D$ the map F_z sending $h \in \mathcal{M}$ to $z \cdot h$ is, as always in this setting, a homeomorphism of $\mathcal{M}$ onto $\mathcal{M}$, and, hence, is an element of the compact Hausdorff space $\mathcal{M}^{\mathcal{M}}$. The closure of the collection of all such maps F_z, $z \in D$, in this latter space is called the *enveloping semigroup*, $E(\mathcal{M})$, of $\mathcal{M}$. One is led by the natural semigroup structure of $E(\mathcal{M})$ to define

$$\phi \cdot h = \lim z_\alpha \cdot h$$

for ϕ, $h \in \mathcal{M}$ and $\{z_\alpha\}$ a net in D tending to ϕ. It is then fairly easy to prove the following result (see [**3**, Chapter 3]).

THEOREM 2. *With the above definition of multiplication in $\mathcal{M}$, $\mathcal{M}$ is (isomorphic and homeomorphic to) the enveloping semigroup $E(\mathcal{M})$ of the transformation group of D acting on $\mathcal{M}$. In particular,*

(a) *The maps $\phi \to \phi \cdot h$, $\phi \to z \cdot \phi$ are continuous for each $h \in \mathcal{M}$, $z \in D$.*

(b) *Thus, for each $h \in \mathcal{M}$ the map τ_h extends continuously to $\mathcal{M}$ and $\tau_h(\mathcal{M})$ is the closure of the orbit of h, i.e., the closure of the part of h.*

(c) *The minimal closed invariant sets of $\mathcal{M}$ coincide with the minimal left ideals of the semigroup $\mathcal{M}$ ($I \subset \mathcal{M}$ is a left ideal if $h \cdot I \subset I$ for all $h \in \mathcal{M}$).*

A subset S of the topological group D is *syndetic* (or *relatively dense*) if there is a compact set $K \subset D$ with $D = K \cdot S$. A point h of the transformation group $(D, \mathcal{M})$ is an *almost periodic point* if for each neighborhood U of h, $\{z\colon\ z \cdot h \in U\}$ is syndetic.

The student of Hoffman's paper [**5**] will immediately recognize the naturalness of the idea of syndetic subsets of D.

LEMMA 1. *A subset S of D is syndetic if and only if some pseudohyperbolic neighborhood of S covers D.*

PROOF. "*If*": Suppose the δ neighborhood of S covers D, $0 < \delta < 1$. Let K be the closure of the pseudohyperbolic neighborhood of radius δ about 0. Let $z_0 \in D$. There is a $w_0 \in S$ such that $\rho(z_0, w_0) < \delta$ or $|z_0 \cdot w_0^{-1}| < \delta$. If $w = z_0 \cdot w_0^{-1}$, then $w \in K$ and $z_0 = w \cdot w_0 \in K \cdot S$ and this direction is clear.

"*Only if*": Suppose $K \subset D$ is compact and $D = K \cdot S$. Choose a pseudohyperbolic disc Δ of radius δ about 0 with $K \subset \Delta$, $0 < \delta < 1$. Then $D = \Delta \cdot S$. If $z_0 \in D$, there are $w_0 \in S$, $u_0 \in \Delta$ such that $z_0 = u_0 \cdot w_0$. Then, $z_0 \cdot w_0^{-1} \in \Delta$. Hence, $\rho(z_0, w_0) = |z_0 \cdot w_0^{-1}| < \delta$ and the result follows.

As Hoffman shows in [**5**], each one point part is in the closure of each subset S of D some neighborhood of which covers D, i.e., each syndetic subset. In particular this is true of points in the Šilov boundary of $\mathcal{M}$ and we obtain

LEMMA 2. *Let $f \in H^\infty(D)$, $S \subset D$ syndetic. If $|f| \leqslant M$ on S, then $|f| \leqslant M$ everywhere on $\mathcal{M}$.*

We next collect results in Ellis [**3**, pp. 10 and 20] and apply them in the present case to obtain

LEMMA 3. *Let $h \in \mathscr{M}$ and let ϕ be a one point part of $\mathscr{M}$. Then, the following statements are equivalent*:

(a) *The closure of the orbit of h is a minimal closed invariant set.*

(b) *h is an almost periodic point.*

(c) *The closure of the orbit h equals $\{\phi \cdot h\} = \{h\}$.*

We are now in a position to state the main theorem of the paper.

THEOREM 3. *Let $h \in \mathscr{M}$. Then the following are equivalent*:

(a) *$\{h\}$ is a trivial part.*

(b) *$\{h\}$ is a minimal closed invariant set.*

(c) *h is an almost periodic point.*

(d) *$\{h\}$ is a minimal left ideal in (the semigroup) $\mathscr{M}$.*

(e) *$h \cdot h = h$.*

(f) *h is the limit of a maximal invariant filter in D.*

Furthermore, the collection of all one point parts of $\mathscr{M}$ is the unique minimal right ideal of $\mathscr{M}$. In addition, the correspondence between maximal invariant filters and one point parts is not one-to-one.

PROOF. We have already made it clear that (a) implies (b) implies (c). Lemma 3 shows that (c) implies (a).

For interest we also supply a proof that (c) implies (a) which does not require the identification of $\mathscr{M}$ as the enveloping semigroup. Let $f \in H^\infty$, $\varepsilon > 0$, and let $U = \{\phi\colon |f(\phi) - f(h)| < \varepsilon\}$. Then U is a neighborhood of h so, since we are assuming that h is an almost periodic point,

$$\begin{aligned}\{z\colon z \cdot h \in U\} &= \{z\colon |f(z \cdot h) - f(h)| < \varepsilon\} \\ &= \{z\colon |(f \circ \tau_h)(z) - (f \circ \tau_h)(0)| < \varepsilon\}\end{aligned}$$

is syndetic. By Lemma 2, $|(f \circ \tau_h)(z) - (f \circ \tau_h)(0)| < \varepsilon$ for all $z \in D$. Since ε was arbitrary, $(f \circ \tau_h)(z) = (f \circ \tau_h)(0)$ for all $z \in D$. Since this is true for all $f \in H^\infty$, the map τ_h is a constant and (a) follow.

The equivalences of (d) and (e) to (a), (b), and (c) follow from the facts that the minimal left ideals of $\mathscr{M}$ coincide with the minimal closed invariant subsets of the transformation group and that each such minimal left ideal contains an idempotent (see [**3**, pp. 17–18]). (Note that (e) gives another way in which a trivial part "equals its square.")

We now know that $\phi \cdot h = h$ for all $\phi \in \mathscr{M}$, h any one point part. Let I be the union of all one point parts. For any $\psi \in \mathscr{M}$, $h \in I$, $\phi \cdot (h \cdot \psi) = (\phi \cdot h) \cdot \psi = h \cdot \psi$ so $h \cdot \psi \in I$. Thus, $I \cdot \mathscr{M} \subset I$ and I is a right ideal. Let M be any right ideal, $h \in I$. Then, $M \supset M \cdot h = \{h\}$ so $h \in M$. Thus, $I \subset M$ showing that I is the unique minimal right ideal in $\mathscr{M}$.

We next show the equivalence of (f) and (b). Again one may follow Ellis [**3**] to extend the action of D on itself to βD, the Stone-Čech compactification of D. Then (using the solution to the Corona problem [**2**]) $\mathscr{M}$ can be viewed both topologically and as a semigroup as the quotient space obtained from βD by identifying ultrafilters on which all $f \in H^\infty(D)$ agree. The key to the proof is the proposition [**3**, p. 72] that a nonempty closed invariant subset of βD is a minimal closed invariant set if and only if it is the adherence of a maximal invariant filter on D.

Suppose that $h \in \mathcal{M}$ is the limit of a maximal invariant filter on D. This latter filter then has a minimal closed invariant subset, M, of βD as its adherence. Then, if p: $\beta D \to \mathcal{M}$ is the natural projection, $p(M) = \{h\}$. Since p is a homomorphism, $\{h\}$ is also a minimal closed invariant subset of $\mathcal{M}$ and thus $\{h\}$ is a trivial part.

Conversely, suppose $\{h\}$ is a trivial part of $\mathcal{M}$. Choose $\phi \in \beta D$ with $p(\phi) = h$. Then, for every $z \in D$, $p(z \cdot \phi) = z \cdot h = h$. By continuity the closed orbit of ϕ projects to h. Since every closed invariant set contains a minimal such, we see that some minimal closed invariant set, M, in βD projects to h. Therefore, h is the limit of the maximal invariant filter whose adherence is M.

Finally, to prove the last comment in the theorem we will show the existence of a point h (in the Šilov boundary) of $\mathcal{M}$ which is the limit of two different maximal invariant filters.

We claim there is a function F on D which is uniformly continuous in the pseudohyperbolic metric on D, $F(re^{i\theta}) = F(r)$ for all $0 \leqslant r < 1$ and all θ, $0 \leqslant F(z) \leqslant 1$, and whenever $z_n \to e^{i\theta_0}$ in such a way that $F(z_n) \to c$ one also has $F(z \cdot z_n) \to c$ for each $z \in D$ but whose radial cluster sets are always $[0,1]$. For example, choose an increasing sequence $\{x_n\}$ of positive real numbers such that $x_{n+1} - x_n$ increases to ∞. Define g on $[0,\infty)$ by $g(x_{2n}) = 0$, $g(x_{2n+1}) = 1$ all n, and in between such consecutive points extend g to be linear. Then, for $0 \leqslant y_1 < y_2$ and n the largest integer such that $x_{n+1} \leqslant y_2$ we claim that

$$|g(x_2) - g(y_1)| \leqslant \frac{y_2 - y_1}{x_{n+1} - x_n}.$$

If $x_n \leqslant y_1$, this is clear as the slope in absolute value of any secant of the "saw tooth" between x_n and x_{n+2} never exceeds the larger absolute slope of the two lines involved. If $y_1 < x_n$, choose w_1, w_2 in $[x_n, x_{n+1}]$ with $g(w_i) = g(y_i)$, $i = 1, 2$. Then

$$|g(y_2) - g(y_1)| = |g(w_2) - g(w_1)| = \frac{|w_2 - w_1|}{x_{n+1} - x_n} \leqslant \frac{|y_2 - y_1|}{x_{n+1} - x_n}.$$

The map h: $[0,1) \to [0,\infty)$ given by $h(r) = \frac{1}{2}\log((1+r)/(1-r))$ is an isometry when the domain is supplied with the (true) hyperbolic distance and the range with euclidean distance. Define $F(re^{i\theta}) = F(r) = g(h(r))$. Letting $r_n = h^{-1}(x_n)$, denoting the hyperbolic metric by χ, and using the simple fact that $\chi(re^{i\theta}, \rho e^{i\phi}) \geqslant \chi(r, \rho)$ we obtain

$$\left|F\left(t_2 e^{i\theta}\right) - F\left(t_1 e^{i\phi}\right)\right| = |F(t_2) - F(t_1)| \leqslant \frac{\chi(t_2, t_1)}{x_{n+1} - x_n} \leqslant \frac{\chi\left(t_2 e^{i\theta}, t_1 e^{i\phi}\right)}{x_{n+1} - x_n}$$

for $0 \leqslant t_1 < t_2$, any θ and ϕ, and with n defined as above by $y_2 = h(t_2)$. This easily shows that F is uniformly continuous in either the hyperbolic or pseudohyperbolic metric. Let z_k converge to a point of T with $F(z_k) \to c$ and fix $z \in D$. Then, $z \cdot z_k$ converges to the same point and $\rho(z_k, z \cdot z_k) = \rho(0, z) = |z|$, so $\chi(z_k, z \cdot z_k)$ is bounded. With $n = n(k)$ determined from z_k and $z \cdot z_k$ as above we get

$$|F(z_k) - F(z \cdot z_k)| \leqslant \frac{\chi(z_k, z \cdot z_k)}{x_{n+1} - x_n} \to 0$$

as $k \to \infty$ completing the verification of the properties of F.

Thinking of F as a function on βD, the uniform continuity shows that F is continuously extendable also to each point in $\mathscr{M}$ which lies in a nontrivial part. We see that F is constant on each such part. Choose two radial parts P_1 and Q_1 at 1 such that $F = 0$ on P_1 and $F = 1$ on Q_1. Transferring completely to $\mathscr{M}$ choose one point parts $h_1(1)$, $h_2(1)$ in the closures of P_1 and P_2, respectively. Using the rotational symmetry of F similarly locate one point parts $h_1(e^{i\theta})$, $h_2(e^{i\theta})$ at $e^{i\theta} \in T$. For each $e^{i\theta} \in T$ choose maximal invariant filters $\mathscr{F}_1(e^{i\theta})$, $\mathscr{F}_2(e^{i\theta})$ converging to $h_1(e^{i\theta})$, $h_2(e^{i\theta})$, respectively. Going back to βD we see that $F = 0$ on the adherence of $\mathscr{F}_1(e^{i\theta})$ and $F = 1$ on the adherence of $\mathscr{F}_2(e^{i\theta})$.

Choose an ultrafilter $\mathscr{G}$ on T which has no set of zero measure as an element. (This is the way each point in the Šilov boundary of $\mathscr{M}$, alias the maximal ideal space of L^∞, may be determined.) We devise the filter $\mathscr{H}_1$ to consist of all subsets, H, of D of the form

$$H = \bigcup_{e^{i\theta} \in G \in \mathscr{G}} v(e^{i\theta}),$$

where v is any selection for each $e^{i\theta} \in T$ of an element $v(e^{i\theta}) \in \mathscr{F}_1(e^{i\theta})$. It is routine to check that $\mathscr{H}_1$ is a filter and, since each $\mathscr{F}_1(e^{i\theta})$ is invariant, that $\mathscr{H}_1$ is an invariant filter on D. Then, as we clearly may, we choose a maximal invariant filter $\mathscr{S}_1$ containing $\mathscr{H}_1$. We similarly obtain a maximal invariant filter $\mathscr{S}_2$ using the above construction with the filters $\mathscr{F}_2(e^{i\theta})$. Clearly, $F = 0$ on the adherence of $\mathscr{S}_1$ while $F = 1$ on the adherence of $\mathscr{S}_2$. Thus, $\mathscr{S}_1$ and $\mathscr{S}_2$ are different maximal invariant filters on D.

However, if $f \in H^\infty$ then the subset of T for which f fails to have nontangential limits is of zero measure and does not belong to $\mathscr{G}$. So, if $e^{i\theta} \in G \in \mathscr{G}$, then f has the same constant value at $h_1(e^{i\theta})$ as at $h_2(e^{i\theta})$, i.e., the same limit on each of $\mathscr{F}_1(r^{i\theta})$ and $\mathscr{F}_2(e^{i\theta})$. This forces f to have the same limit on each of $\mathscr{S}_1$ and $\mathscr{S}_2$. Thus, the distinct filters $\mathscr{S}_1$ and $\mathscr{S}_2$ converge to the same trivial part (in the Šilov boundary) as required.

3. Comments. If the correspondence between maximal invariant filters and trivial parts had turned out to be unique there is a sense in which one could have claimed the investigation of parts of $\mathscr{M}$ to have been completed. There are easier ways to use the function F of the above proof of Theorem 3 to show that the correspondence is not unique. The reason we gave the proof we did is to give clues as to why the correspondence is not unique.

Here is the picture which emerges in our minds from the proof of Theorem 3 adapted intuitively to nets rather than filters. First, we see that any net converging to a one point part must get so thoroughly mixed up with its group translates that bounded analytic functions cannot distinguish these different translated nets. However, things are worse than this. The different nets just referred to at least are hyperbolically close. However, in the construction, for example, we could have chosen the points $h_1(e^{i\theta})$ and $h_2(e^{i\theta})$ to be tangential, one on one side of $e^{i\theta}$ and the other on the other side. In a sense the nets corresponding to the filters $\mathscr{S}_1$ and $\mathscr{S}_2$ are "infinitely" far apart and yet tend to the same homomorphism. Looking at the construction we see that this "strange" phenomenon actually occurs because of a "nice" property of $H^\infty(D)$ associated with analyticity, namely, the Fatou theorem assuring existence of nontangential limits almost everywhere.

The question we pose is this. Can the identification of appropriate maximal invariant filters be "explained" completely by such well-known properties of $H^\infty(D)$?

REFERENCES

1. T. J. Abram, *Parts in the maximal ideal space of H^∞—A harmonic analysis approach*, Doctoral Dissertation, University of California, Santa Barbara, 1983, pp. 1–94.

2. L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 542–559.

3. R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969, pp. 1–211.

4. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 1–217.

5. ______, *Bounded analytic functions and Gleason parts*, Ann. of Math. (2) **81** (1967), 74–111.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, STANISLAUS, TURLOCK, CALIFORNIA 95380

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106