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ON THE RADON-NIKODÝM PROPERTY IN JORDAN TRIPLES

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ABSTRACT. Given a JBW*-triple U, we determine when its predual U_* and dual U^* possess the Radon-Nikodým property.

1. Introduction. Recently several authors [**1**, **5**, **6**, **7**, **8**, **9**] have obtained significant results concerning the JB*-triples which were introduced by Kaup [**9**] who has shown that every bounded symmetric domain in a complex Banach space is biholomorphic to the unit ball of a JB*-triple. In this note, we show that the predual of a JBW*-triple U possesses the Radon-Nikodým property if, and only if, U is an l_∞-sum of Cartan factors. We also show that the dual of a JBW*-triple U possesses the Radon-Nikodým property if, and only if, U is a finite sum of certain Cartan factors.

A JB*-*triple* is a complex Banach space U together with a continuous, sesquilinear mapping D: $U \times U \to L(U)$, where $L(U)$ is the algebra of bounded linear operators on U, such that for $x, y, u, v, z \in U$, the following conditions are satisfied:

(i) $[D(x,y), D(u,v)] = D(D(x,y)u,v) - D(u, D(v,x)y)$;

(ii) $\|D(z,z)\| = \|z\|^2$;

(iii) $D(z,z)$ is a positive hermitian operator,

where the operation [,] in (i) denotes the Lie product.

A JBW*-*triple* is a JB*-triple U which has a (necessarily unique) predual U_*.

Let H and K be complex Hilbert spaces. As in [**8**], we define the *Cartan factors* of types 1 to 6 as follows:

$C^1 = L(H,K)$ with dim $K \leq$ dim H;

$C^2_n = \{z \in L(H)\colon\ z = -jz^*j\}$, where j: $H \to H$ is a conjugation, i.e., j is an antilinear involutive isometry and $n = \dim H \geq 4$;

$C^3_n = \{z \in L(H)\colon\ z = jz^*j\}$;

C^4 = any selfadjoint closed subspace U of $L(H)$ of dimension ≥ 3 such that $z^2 \in \mathbf{C}I$ for all z in U;

$C^5 = M_{1,2}(\mathbf{O})$—the 1×2 matrices over the Cayley algebra $\mathbf{O}$;

$C^6 = M_3(\mathbf{O})_{sa}$—the selfadjoint 3×3 matrices over $\mathbf{O}$.

We note that C^2_n and C^3_n consist of the 'antisymmetric' and 'symmetric' operators respectively and that the 'spin factor' C^4 is a reflexive Banach space. Evidently, C^5 and C^6 are finite-dimensional. We refer to [**4**] for the basics concerning the Radon-Nikodým property in Banach spaces.

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2. Main results.

LEMMA 1. *The predual of a Cartan factor C has the Radon-Nikodým property.*

PROOF. Since reflexive spaces have the Radon-Nikodým property [**4**, p. 76], we need only consider the case when C is of type 1, 2, or 3. If C is of type 1, then $C = L(H, K)$; but the predual $L(H, K)_*$ can be identified with the projective tensor product $H\hat{\otimes}\overline{K}$, where $\overline{K}$ is the conjugate of K, and $H\hat{\otimes}\overline{K}$ has the Radon-Nikodým property [**4**, p. 249]. If C is of type 3, then $C = \{z \in L(H)\colon z = jz^*j\}$ with conjugation $j\colon H \to H$. Let $T(H)$ be the Banach space of trace-class operators on H. Then $T(H)$ has the Radon-Nikodým property and so does its subspace $N = \{t \in T(H)\colon t = jt^*j\}$. Furthermore, one can identify C with the Banach dual N^* of N via the duality $(z,t) \in C \times N \mapsto \mathrm{Tr}(zt)$, where Tr denotes the canonical trace (cf. [**2**, Proposition 1]). Since the predual C_* of C is unique, it follows that C_* is isometric to N and hence C_* has the Radon-Nikodým property. Likewise, if $C = \{z \in L(H)\colon z = -jz^*j\}$, then C_* also has the Radon-Nikodým property.

THEOREM 2. *Let U be a JBW*-triple. Then its predual U_* has the Radon-Nikodým property if, and only if, U is an l_∞-sum of Cartan factors.*

PROOF. Suppose U_* has the Radon-Nikodým property, then U_* has the Krein-Milman property [**4**, p. 190] and so its unit ball U_*^1 is the norm closed convex hull of its extreme points. Hence U_* coincides with the norm closure of the linear span of the extreme points of U_*^1 and by [**6**, Theorems 1 and 2], U is the $\sigma(U, U_*)$-closure of the linear span of its minimal tripotents, that is, U is atomic and hence U is an l_∞-sum of Cartan factors (cf. [**7**, Proposition 2] and [**8**, 9.1.2]). The converse follows from Lemma 1.

REMARK. The above result implies that if the predual of a JBW*-triple U has the Radon-Nikodým property, then $U = V^{**}$, where V is a JB*-triple ideal in V^{**}. This result has also been obtained by Li [**10**].

COROLLARY 3. *Let U be a JB*-triple. Then U^* has the Radon-Nikodým property if, and only if, U^{**} is an l_∞-sum of Cartan factors.*

We remark that Theorem 2 has been obtained recently and independently by Barton and Gedefroy in [**13**] where they have also shown that the dual of a JB*-triple U has the Radon-Nikodým property if U does not contain an isomorphic copy of l_1. We thank the referee for drawing our attention to the results of Barton and Godefroy.

THEOREM 4. *Let U be a JBW*-triple. Then its dual U^* has the Radon-Nikodým property if, and only if, U is a finite direct sum of any of the following Cartan factors: $L(H,K)$, C_n^2, C_n^3, C^4, C^5, C^6 with n, $\dim K < \infty$.*

PROOF. Let U^* have the Radon-Nikodým property. Then the predual U_* has the Radon-Nikodým property. So there is a family $\{e_\alpha\}$ of minimal tripotents in U such that $U = \bigoplus_\alpha J_\alpha$ (l_∞-sum), where J_α is the intersection of all $\sigma(U, U_*)$-closed ideals containing e_α and J_α is a Cartan factor (cf. [**7**, Proposition 2]). If $\{e_\alpha\}$ is infinite, then its closed linear span contains a copy of l_∞ and since U^* has the Radon-Nikodým property, this is impossible. So $\{e_\alpha\}$ is finite and U is a finite sum of Cartan factors J_α. If J_α is C_n^3 with $J_\alpha = \{z \in L(H)\colon z = jz^*j\}$ for some conjugation $j\colon H \to H$, then the selfadjoint part A of J_α is a JW-factor which is not a spin factor and, also, the von Neumann algebra generated by A is $L(H)$. As A^* has the Radon-Nikodým property, by [**3**, Lemmas 6 and 4], $L(H)$ is of type I_m with $m < \infty$. So $n = \dim H = m < \infty$. Next, if J_α is C_n^2 with $J_\alpha = \{z \in L(H)\colon\ z = -jz^*j\}$ and if H is infinite dimensional, then J_α contains a unitary operator u such that the map $z \in J_\alpha \mapsto u^*z \in u^*J_\alpha$ is a surjective linear isometry and, also, the selfadjoint part A of u^*J_α is a JW-factor [**12**, p. 333]. Since $(J_\alpha)^*$ has the Radon-Nikodým property, A^* also has the property and as before, the von Neumann algebra $L(H)$ generated by A is of type I_m with $m < \infty$ which is impossible. So $n = \dim H < \infty$. Finally, if J_α is C^1, then $J_\alpha = L(H, K)$ with K a subspace of H. Now $L(H,K)^* = (H \hat{\otimes} \overline{K})^{**}$ has the Radon-Nikodým property and since there is a norm 1 projection from H onto K, $K \hat{\otimes} \overline{K}$ can be embedded into $H \hat{\otimes} \overline{K}$ [**11**, Theorem 3.10] and it follows that $L(K)^* = (K \hat{\otimes} \overline{K})^{**}$ has the Radon-Nikodým property which implies $\dim K < \infty$. This completes the proof.

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