GENERALIZATIONS OF PALEY-WIENER'S THEOREM FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE
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ABSTRACT. An interpretation and generalizations of the Paley-Wiener theorem for entire functions of exponential type are given in connection with the Fourier-Laplace transform.

1. Introduction. The important generalization of the Paley-Wiener theorem [6] for entire functions of exponential type was given by Plancherel and Pólya [7] (see Fuks [3] and Ronkin [8]). A further extension of Plancherel and Pólya was given by Martin [5] in the case of functions $f$ analytic on the octant $\text{Im} \ z_k > 0$ ($k = 1, 2, \ldots, n$). In this paper, we discuss Martin's theorem in a general situation following the idea of the general theory [9, 10, 11] of integral transforms.

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2. Fourier-Laplace transform and a fundamental problem. We set

$$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n,$$
$$z_j = x_j + iy_j \quad (x_j, y_j \in \mathbb{R}),$$
$$x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n,$$
$$dx = dx_1 dx_2 \cdots dx_n, \quad dy = dy_1 dy_2 \cdots dy_n,$$
$$t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n, \quad (z, t) = \sum_{j=1}^{n} z_j t_j.$$

For some domains $D \subset \mathbb{R}^n$, $\Omega \subset \mathbb{C}^n$, and for functions $F$ of $L_2(D, dt)$, we consider the Fourier-Laplace transform

$$f(z) = \left(\frac{1}{2\pi}\right)^{n/2} \int_D F(t) e^{-i(z, t)} dt.$$  \hfill (2.1)

When we consider the images $f(z)$ of $L_2(D, dt)$ functions $F$ by (2.1), we should consider the following expression:

$$K(z, \bar{w}; \Omega, D) = \left(\frac{1}{2\pi}\right)^{n} \int_D e^{-i(z, t)} e^{i(\bar{w}, t)} dt.$$  \hfill (2.2)

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See the general theory [9, 10, 11] of integral transforms for this idea. Thus, we first assume that

\[(2.3) \int_D e^{2(y,t)} \, dt < \infty \quad \text{on } \Omega.\]

Note that (2.2) exists on $\Omega \times \overline{\Omega}$ if and only if (2.3) is valid and, further, the existence domain of $K(z,\overline{w};\Omega, D)$ is independent of $x$. Hence, we can naturally consider $\Omega$ as a tube domain of the form $T_G = R^n + iG \subset C^n$. Moreover, we set

\[(2.4) \hat{G}_D = \left\{ y \in R^n, \int_D e^{2(y,t)} \, dt < \infty \right\}\]

and we can consider the maximal domain $T_{G_D}$ as $\Omega$. Of course, $\hat{G}_D$ is a convex domain on $R^n$. We thus consider the function $K(z,\overline{w};T_{G_D}, D)$ on $T_{G_D} \times T_{\overline{G_D}}$. This function is a positive matrix on $T_{G_D}$ in the sense of E. M. Moore and so there exists a uniquely determined Hilbert space $H_K(T_{G_D}, D)$ composed of functions on $T_{G_D}$ admitting the reproducing kernel $K(z,\overline{w};T_{G_D}, D)$ (see Aronszajn [1, 2]). This space $H_K(T_{G_D}, D)$ is composed of holomorphic functions $f(z)$ on $T_{G_D}$ which are expressible in the form

\[(2.5) f(z) = \left( \frac{1}{2\pi} \right)^{n/2} \int_D F(t)e^{-i(z,t)} \, dt\]

for $L_2(D,dt)$ functions $F$ and with the norm

\[(2.6) \|f\|^2_{H_K(T_{G_D}, D)} = \int_D |F(t)|^2 \, dt.\]

[9, 10, 11]. On the other hand, from Parseval’s equation we have

\[(2.7) \int_{R^n} |f(x)|^2 \, dx = \int_D |F(t)|^2 \, dt.\]

Hence,

\[(2.8) \|f\|^2_{H_K(T_{G_D}, D)} = \int_{R^n} |f(x)|^2 \, dx.\]

We thus see that the functions $f(z)$ of $H_K(T_{G_D}, D)$ are analytic on $T_{G_D}$, $L_2(R^n, dx)$-integrable, and the norms of $f$ in $H_K(T_{G_D}, D)$ are given by (2.8). In this situation, we can regard that when $D$ is a bounded interval in $R$ and when $D$ is a bounded convex domain in $R^n$, the theorems of Paley-Wiener [6] and Plancherel-Pólya [7] give characterizations of the members $f(z)$ of $H_K(T_{G_D}, D)$ in terms of the growth of $f(z)$ at infinity, respectively. We thus, in general, propose a fundamental problem in our situation.

A fundamental problem. In the above situation, give a characterization of the members of $H_K(T_{G_D}, D)$ in terms of the domain $D$.

In order to give a reasonable solution for this problem, we will assume that $D$ is a convex domain and $\partial D$ is a smooth hypersurface on $R^n$. When $D = \prod_{j=1}^n (-\infty, a)$ ($a > 0$), Martin [5] discussed the growth of the functions of $H_K(T_{G_D}, D)$ at infinity, but he did not give a complete answer for the above problem in his situation.
3. On ranges. Let $O$ be the origin of coordinates in the $t$-space $R^n$ and let the hyperplanes $\{\Gamma\}$ pass through it and lie parallel to the limiting positions of the tangent hyperplanes of $\partial D$. We consider the convex cone with vertex at the origin enveloped by these hyperplanes. The nappe of this cone lying on the same side of the hyperplanes $\{\Gamma\}$ as the domain $D$ is called the asymptotic cone of $TD$. As the asymptotic cone of a bounded domain $D$, we take the set $\{0\}$; that is, the origin. When $V$ is the asymptotic cone of $TD$, we will say that the domain $TD$ is of type $V$.

We consider the conjugate cone $V^*$ of $V$; that is,

$$V^* = \left\{ (t_1^*, t_2^*, \ldots, t_n^*) \in R^n; \sum_{j=1}^{n} t_j^* t_j > 0 \text{ for all } t \in V, t \neq 0 \right\}.$$

When $D$ contains a whole line, $V^*$ does not contain any $n$-dimensional sphere. Further then, since $\hat{G}_D = \{0\}$, in the sequel we assume that $D$ does not contain any whole line.

We will consider $V^*$ in the $y$-space $R^n$. Then, note that

$$e^{(y,t)} \text{ is bounded on } t \in D$$

if and only if

$$-y \in V^* \text{ or } y \in -V^*.$$

We define the support function of the convex set $\overline{D}$ by

$$H_D(y) = \max_{t \in \overline{D}} (y,t).$$

Then we obtain

**Theorem 3.1.** $\hat{G}_D = -V^*$.

**Proof.** For any fixed point $y^{(0)} \in -V^*$, we set

$$(y^{(0)}, t) = |y^{(0)}| |t| \cos \theta_0(t) < 0 \quad \text{on } V,$$

where, of course, $\theta_0(t)$ ($|\theta_0(t)| \leq \pi$) is the angle between the two vectors $y^{(0)}$ and $t$ in the same space $R^n$. Hence, there exists $\Theta$ such that

$$|\theta_0(t)| \geq \Theta > \pi/2 \quad \text{on } V.$$

Hence, there exist $\varepsilon > 0$ and $M > 0$ such that

$$|\theta_0(t)| \geq \Theta - \varepsilon > \pi/2 \quad \text{on } D \cap \{|t| \geq M\}.$$

Then, from the identity

$$\int_{|x| < N} f(|x|) \, dx = \frac{2\sqrt{\pi^n}}{\Gamma(n/2)} \int_0^N x^{n-1} f(x) \, dx$$
[4, p. 623], we have

\[
\int_D e^{2(y(0),t)} dt \leq \int_{D \cap \{|t| \leq M\}} e^{2(y(0),t)} dt + \int_{D \cap \{|t| \geq M\}} e^{2|y(0)| |t| \cos(\Theta-\epsilon)} dt
\]

\[
\leq \int_{D \cap \{|t| \leq M\}} e^{2(y(0),t)} dt + \int_{\mathbb{R}^n} e^{2|y(0)| |t| \cos(\Theta-\epsilon)} dt
\]

(3.6)

\[
\leq \int_{D \cap \{|t| \leq M\}} e^{2(y(0),t)} dt + \lim_{N \to \infty} \frac{2\sqrt{\pi^n}}{\Gamma(n/2)} \int_0^N t^{n-1} e^{t|2y(0)| \cos(\Theta-\epsilon)} dt < \infty.
\]

Hence, we have \( \hat{G}_D \supset -V^* \).

On the other hand, for any point \( y(0) \in (-V^*)^c \), the complement, by the definition of \( V^* \) there exists a point \( t(0) \in V \) such that \( (y(0),t(0)) > 0 \). Then, there exists a narrow nondegenerate (i.e. contains an \( n \)-dimensional sphere) convex cone \( \Gamma(t(0)) \) with vertex 0 such that

\[
(y(0),t) > 0 \quad \text{on} \quad \Gamma(t(0))
\]

(3.7)

and

\[
D \supset \Gamma(t(0)) \cap \{|t| \geq M\} \quad \text{for some} \quad M > 0.
\]

Then

\[
\int_D e^{2(y(0),t)} dt \geq \int_{\Gamma(t(0)) \cap \{|t| \geq M\}} e^{2(y(0),t)} dt
\]

(3.9)

\[
\geq \int_{\Gamma(t(0)) \cap \{|t| \geq M\}} dt = \infty.
\]

Hence, \( \hat{G}_D \subset -V^* \), and we have the desired result.

In Theorem 3.1, we will give a characterization of the functions \( f \in H_K(T_{-V^*}, D) \).

4. Necessity condition. We set

\[
y = \rho \lambda \quad (y_j = \rho \lambda_j, \quad \rho > 0) \quad \text{and} \quad |\lambda| = 1.
\]

From the identity (2.5), we have, by Parseval's equation,

\[
\int_{\mathbb{R}^n} |f(x + i\rho \lambda)|^2 dx = \int_D |F(t)|^2 e^{2\rho H(\lambda,t)} dt
\]

(4.1)

\[
\leq e^{2\rho H_D(\lambda)} \int_D |F(t)|^2 dt.
\]

Hence, for \( \rho \lambda \in -V^* \) we have

\[
\frac{1}{2} \lim_{\rho \to -\infty} \frac{1}{\rho} \log \int_{\mathbb{R}^n} |f(x + i\rho \lambda)|^2 dx \leq H_D(\lambda).
\]

(4.2)

Further, when \( f(t) \neq 0 \) a.e. on \( D \) we can prove that the actual limit exists and that it is equal to \( H_D(\lambda) \) as in Martin [5].
5. Sufficiency condition. In our situation, we will give a complete answer for our fundamental problem:

**Theorem 5.1.** \( f(z) \) belongs to \( H_K(T_-V^*, D) \) if and only if

\[ f(z) \text{ is holomorphic on } T_-V^* \text{ and } L_2(R^n, dx) \text{-integrable, and further, for any } y \in -V^*, \text{ the integral} \]

\[ \int_{R^n} |f(x + iy)|^2 \, dx \]

exists,

and

\[ \frac{1}{2} \lim_{\rho \to \infty} \frac{1}{\rho} \log \int_{R^n} |f(x + i\rho \lambda)|^2 \, dx \leq H_D(\lambda). \]

**Proof.** It is sufficient to prove that any function \( f(z) \) satisfying (5.1) and (5.2) is an image by the Fourier-Laplace transform (2.1) of an \( L_2(D, dt) \) function. Since \( f(x) \in L_2(R^n, dx) \), we can define the \( L_2(R^n, dt) \) function \( \tilde{F}(t) \) by

\[ (5.3) \quad \tilde{F}(t) = \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^{n/2} \int_{|x_j| < N} f(x_j) e^{ix_j t} \, dx. \]

Of course,

\[ (5.4) \quad f(x) = \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^{n/2} \int_{|t_j| < N} \tilde{F}(t_j) e^{-ix_j t} \, dt, \]

in the framework of \( L_2 \) spaces.

We first assume that in addition \( f(x) \in L_1(R^n, dx) \). Then, (5.3) exists in the ordinary sense

\[ \tilde{F}(t) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{R^n} f(x) e^{ix \cdot t} \, dx. \]

By condition (5.2), since the integrals \( \int_{R^n} |f(x + iy)|^2 \, dx \) exist for all \( y \in -V^* \), the integrals

\[ (5.5) \quad \left( \frac{1}{2\pi} \right)^{n/2} \int_{R^n} f(x + iy) e^{ix + iy \cdot t} \, dx \]

also exist for all \( y \in -V^* \). Moreover, by using the Cauchy integral theorem, we see that the integrals (5.5) are independent of \( y \in -V^* \). Hence, we set

\[ \tilde{F}(t) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{R^n} f(x + iy) e^{ix + iy \cdot t} \, dx. \]

(See [12, pp. 98–101] for this argument.) Then, we see immediately that \( \tilde{F}(t) \) is continuous on \( R^n \) and \( \tilde{F}(t) = \hat{F}(t) \) on \( R^n \). Hence, by Parseval’s equation, we have

\[ \int_{R^n} |f(x + iy)|^2 \, dx = \int_{R^n} |\tilde{F}(t)|^2 e^{2iy \cdot t} \, dt. \]
Hence, for \( y = \rho \lambda \in -V^* \), we have

\[
H_D(\lambda) \geq \frac{1}{2} \lim_{\rho \to \infty} \frac{1}{\rho} \log \int_{\mathbb{R}^n} |f(x + i\rho \lambda)|^2 \, dx
\]

(5.6)

\[
= \frac{1}{2} \lim_{\rho \to \infty} \frac{1}{\rho} \log \int_{\mathbb{R}^n} |\tilde{F}(t)|^2 e^{2\rho(H(\lambda,t))} \, dt.
\]

For any \( t_0 \in \overline{D}^c \), we will show that \( \tilde{F}(t_0) = 0 \). Since \( \tilde{F}(t) \) is continuous on \( \mathbb{R}^n \), when \( \tilde{F}(t_0) \neq 0 \), for some closed sphere \( S(t_0) \) with the center \( t_0 \ (\subset \overline{D}^c) \),

\[
|\tilde{F}(t)| \geq m > 0 \quad \text{on } S(t_0)
\]

(5.7)

Further then, since

\[
(\lambda, t) > H_D(\lambda) \quad \text{on } S(t_0),
\]

there exists \( \varepsilon > 0 \) such that

\[
(\lambda, t) \geq H_D(\lambda) + \varepsilon \quad \text{on } S(t_0).
\]

Hence, from (5.6) we have

\[
H_D(\lambda) \geq \frac{1}{2} \lim_{\rho \to \infty} \frac{1}{\rho} \log \left( \int_{S(t_0)} |\tilde{F}(t)|^2 e^{2\rho(H(\lambda,t))} \, dt \right)
\]

\[
\geq \frac{1}{2} \lim_{\rho \to \infty} \frac{1}{\rho} \log \left( m^2 e^{2\rho(H_D(\lambda)+\varepsilon)} \right) \int_{S(t_0)} \, dt
\]

\[
= H_D(\lambda) + \varepsilon,
\]

which implies a contradiction. Hence, the support of \( \tilde{F} \) is contained in \( \overline{D} \). Further, since (2.3) is valid for any \( y \in -V^* \), in (5.4) we obtain the desired expression

\[
f(z) = \left( \frac{1}{2\pi} \right)^n \int_D \tilde{F}(t) e^{-iz(t,t)} \, dt.
\]

When \( f(x) \) is in \( L_2(\mathbb{R}^n, dx) \) but not necessarily in \( L_1(\mathbb{R}^n, dx) \) we set

\[
f_\varepsilon(z) = f(z) \prod_{j=1}^{n} \frac{\sin \varepsilon z_j}{\varepsilon z_j}.
\]

Then we have \( f_\varepsilon \in L_1(\mathbb{R}^n, dx) \). Hence, the Fourier transform \( \tilde{F}_\varepsilon(t) \) of \( f_\varepsilon(z) \) vanishes outside the convex set \( \overline{D} \). From the relation

\[
\tilde{F}_\varepsilon(t) = \left( \frac{1}{2\varepsilon} \right)^n \int_{t_1-\varepsilon}^{t_1+\varepsilon} \cdots \int_{t_n-\varepsilon}^{t_n+\varepsilon} \tilde{F}(t) \, dt
\]

we obtain

\[
\lim_{\varepsilon \to 0} \tilde{F}_\varepsilon(t) = \tilde{F}(t)
\]

for almost all \( t \in \mathbb{R}^n \). Hence, the support of \( \tilde{F} \) is contained in \( \overline{D} \). We thus complete the proof of the theorem.

**ADDED IN PROOF.** In Theorem 5.1, \( f(x) \) are also considered as the boundary values such that

\[
\lim_{y \to 0} f(x + iy) = f(x)
\]

in the sense of the \( L_2 \) norm (see [12, Chapter III]).
GENERALIZATIONS OF PALEY-WIENER’S THEOREM

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