PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

GENERALIZATIONS OF PALEY-WIENER'S THEOREM FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

SABUROU SAITOH

ABSTRACT. An interpretation and generalizations of the Paley-Wiener theorem for entire functions of exponential type are given in connection with the Fourier-Laplace transform.

1. Introduction. The important generalization of the Paley-Wiener theorem [**6**] for entire functions of exponential type was given by Plancherel and Pólya [**7**] (see Fuks [**3**] and Ronkin [**8**]). A further extension of Plancherel and Pólya was given by Martin [**5**] in the case of functions f analytic on the octant $\operatorname{Im} z_k > 0$ $(k = 1, 2, \ldots, n)$. In this paper, we discuss Martin's theorem in a general situation following the idea of the general theory [**9**, **10**, **11**] of integral transforms.

The author wishes to thank Professor T. G. Genchev for pleasant and fruitful discussions in Bulgaria.

2. Fourier-Laplace transform and a fundamental problem. We set

$$\begin{gathered} z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n, \\ z_j = x_j + iy_j \qquad (x_j, y_j \in R), \\ x = (x_1, x_2, \ldots, x_n), \qquad y = (y_1, y_2, \ldots, y_n) \in R^n, \\ dx = dx_1 dx_2 \cdots dx_n, \qquad dy = dy_1 dy_2 \cdots dy_n, \\ t = (t_1, t_2, \ldots, t_n) \in R^n, \qquad (z, t) = \sum_{j=1}^{n} z_j t_j. \end{gathered}$$

For some domains $D \subset R^n$, $\Omega \subset \mathbf{C}^n$, and for functions F of $L_2(D, dt)$, we consider the Fourier-Laplace transform

$$f(z) = \left(\frac{1}{2\pi}\right)^{n/2} \int_D F(t) e^{-i(z,t)}\, dt. \tag{2.1}$$

When we consider the images $f(z)$ of $L_2(D, dt)$ functions F by (2.1), we should consider the following expression:

$$K(z, \overline{w}; \Omega, D) = \left(\frac{1}{2\pi}\right)^{n} \int_D e^{-i(z,t)} e^{i(\overline{w},t)}\, dt. \tag{2.2}$$

Received by the editors December 18, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32A07, 42B10, 44A10; Secondary 30C40.

Key words and phrases. Paley-Wiener's theorem, entire function of exponential type, Fourier-Laplace transform, general theory of integral transforms, reproducing kernel, tube domain, convex domain.

Research partially supported by Grant-in-Aid for Scientific Research 60540086.

See the general theory [**9**, **10**, **11**] of integral transforms for this idea. Thus, we first assume that

$$\int_D e^{2(y,t)}\,dt < \infty \quad \text{on } \Omega. \tag{2.3}$$

Note that (2.2) exists on $\Omega\times\overline{\Omega}$ if and only if (2.3) is valid and, further, the existence domain of $K(z,\overline{w};\Omega,D)$ is independent of x. Hence, we can naturally consider Ω as a tube domain of the form $T_G = R^n + iG \subset \mathbf{C}^n$. Moreover, we set

$$\hat{G}_D = \left\{ y \in R^n, \int_D e^{2(y,t)}\,dt < \infty \right\} \tag{2.4}$$

and we can consider the maximal domain $T_{\hat{G}_D}$ as Ω. Of course, $\hat{G}_D$ is a convex domain on R^n. We thus consider the function $K(z,\overline{w};T_{\hat{G}_D},D)$ on $T_{\hat{G}_D}\times\overline{T_{\hat{G}_D}}$. This function is a positive matrix on $T_{\hat{G}_D}$ in the sense of E. M. Moore and so there exists a uniquely determined Hilbert space $H_K(T_{\hat{G}_D},D)$ composed of functions on $T_{\hat{G}_D}$ admitting the reproducing kernel $K(z,\overline{w};T_{\hat{G}_D},D)$ (see Aronszajn [**1**, **2**]). This space $H_K(T_{\hat{G}_D},D)$ is composed of holomorphic functions $f(z)$ on $T_{\hat{G}_D}$ which are expressible in the form

$$f(z) = \left(\frac{1}{2\pi}\right)^{n/2} \int_D F(t) e^{-i(z,t)}\,dt \tag{2.5}$$

for $L_2(D,dt)$ functions F and with the norm

$$\|f\|^2_{H_K(T_{\hat{G}_D},D)} = \int_D |F(t)|^2\,dt \tag{2.6}$$

[**9**, **10**, **11**]. On the other hand, from Parseval's equation we have

$$\int_{R^n} |f(x)|^2\,dx = \int_D |F(t)|^2\,dt. \tag{2.7}$$

Hence,

$$\|f\|^2_{H_K(T_{\hat{G}_D},D)} = \int_{R^n} |f(x)|^2\,dx. \tag{2.8}$$

We thus see that the functions $f(z)$ of $H_K(T_{\hat{G}_D},D)$ are analytic on $T_{\hat{G}_D}$, $L_2(R^n,dx)$-integrable, and the norms of f in $H_K(T_{\hat{G}_D},D)$ are given by (2.8). In this situation, we can regard that when D is a bounded interval in R and when D is a bounded convex domain in R^n, the theorems of Paley-Wiener [**6**] and Plancherel-Pólya [**7**] give characterizations of the members $f(z)$ of $H_K(T_{\hat{G}_D},D)$ in terms of the growth of $f(z)$ at infinity, respectively. We thus, in general, propose a fundamental problem in our situation.

A fundamental problem. In the above situation, give a characterization of the members of $H_K(T_{\hat{G}_D},D)$ in terms of the domain D.

In order to give a reasonable solution for this problem, we will assume that D is a convex domain and ∂D is a smooth hypersurface on R^n. When $D = \prod_{j=1}^n(-\infty,a)$ ($a>0$), Martin [**5**] discussed the growth of the functions of $H_K(T_{\hat{G}_D},D)$ at infinity, but he did not give a complete answer for the above problem in his situation.

3. On ranges. Let O be the origin of coordinates in the t-space R^n and let the hyperplanes $\{\Gamma\}$ pass through it and lie parallel to the limiting positions of the tangent hyperplanes of ∂D. We consider the convex cone with vertex at the origin enveloped by these hyperplanes. The nappe of this cone lying on the same side of the hyperplanes $\{\Gamma\}$ as the domain D is called the asymptotic cone of T_D. As the asymptotic cone of a bounded domain D, we take the set $\{0\}$; that is, the origin. When V is the asymptotic cone of T_D, we will say that the domain T_D is of type V.

We consider the conjugate cone V^* of V; that is,

$$V^* = \left\{ (t_1^*, t_2^*, \dots, t_n^*) \in R^n; \sum_{j=1}^{n} t_j^* t_j > 0 \text{ for all } t \in \overline{V},\, t \neq 0 \right\}.$$

When D contains a whole line, V^* does not contain any n-dimensional sphere. Further then, since $\hat{G}_D = \{\emptyset\}$, in the sequel we assume that D does not contain any whole line.

We will consider V^* in the y-space R^n. Then, note that

$$e^{(y,t)} \text{ is bounded on } t \in D \tag{3.1}$$

if and only if

$$-y \in V^* \quad \text{or} \quad y \in -V^*. \tag{3.2}$$

We define the support function of the convex set $\overline{D}$ by

$$H_D(y) = \max_{t \in \overline{D}} (y, t). \tag{3.3}$$

Then we obtain

THEOREM 3.1. $\hat{G}_D = -V^*$.

PROOF. For any fixed point $y^{(0)} \in -V^*$, we set

$$(y^{(0)}, t) = |y^{(0)}|\, |t| \cos\theta_0(t) < 0 \quad \text{on } V,$$

where, of course, $\theta_0(t)$ ($|\theta_0(t)| \leq \pi$) is the angle between the two vectors $y^{(0)}$ and t in the same space R^n. Hence, there exists Θ such that

$$|\theta_0(t)| \geq \Theta > \pi/2 \quad \text{on } V. \tag{3.4}$$

Hence, there exist $\varepsilon > 0$ and $M > 0$ such that

$$|\theta_0(t)| \geq \Theta - \varepsilon > \pi/2 \quad \text{on } D \cap \{|t| \geq M\}. \tag{3.5}$$

Then, from the identity

$$\int_{|x|<N} f(|x|)\, dx = \frac{2\sqrt{\pi^n}}{\Gamma(n/2)} \int_0^N x^{n-1} f(x)\, dx$$

[**4**, p. 623], we have

$$\begin{aligned}
\int_D e^{2(y^{(0)},t)}\,dt &\le \int_{D\cap\{|t|\le M\}} e^{2(y^{(0)},t)}\,dt + \int_{D\cap\{|t|\ge M\}} e^{2|y^{(0)}|\,|t|\cos(\Theta-\varepsilon)}\,dt \\
&\le \int_{D\cap\{|t|\le M\}} e^{2(y^{(0)},t)}\,dt + \int_{R^n} e^{2|y^{(0)}|\,|t|\cos(\Theta-\varepsilon)}\,dt \\
&\le \int_{D\cap\{|t|\le M\}} e^{2(y^{(0)},t)}\,dt \\
&\quad + \lim_{N\to\infty}\frac{2\sqrt{\pi^n}}{\Gamma(n/2)}\int_0^N t^{n-1}e^{t[2|y^{(0)}|\cos(\Theta-\varepsilon)]}\,dt \\
&< \infty.
\end{aligned} \tag{3.6}$$

Hence, we have $\hat{G}_D \supset -V^*$.

On the other hand, for any point $y^{(0)} \in (-V^*)^c$, the complement, by the definition of V^* there exists a point $t^{(0)} \in V$ such that $(y^{(0)}, t^{(0)}) > 0$. Then, there exists a narrow nondegenerate (i.e. contains an n-dimensional sphere) convex cone $\Gamma(t^{(0)})$ with vertex 0 such that

$$(y^{(0)}, t) > 0 \quad \text{on } \Gamma(t^{(0)}) \tag{3.7}$$

and

$$D \supset \Gamma(t^{(0)}) \cap \{|t| \ge M\} \quad \text{for some } M > 0. \tag{3.8}$$

Then

$$\begin{aligned}
\int_D e^{2(y^{(0)},t)}\,dt &\ge \int_{\Gamma(t^{(0)})\cap\{|t|\ge M\}} e^{2(y^{(0)},t)}\,dt \\
&\ge \int_{\Gamma(t^{(0)})\cap\{|t|\ge M\}} dt = \infty.
\end{aligned} \tag{3.9}$$

Hence, $\hat{G}_D \subset -V^*$, and we have the desired result.

In Theorem 3.1, we will give a characterization of the functions $f \in H_K(T_{-V^*}, D)$.

4. Necessity condition.

We set

$$y = \rho\lambda \quad (y_j = \rho\lambda_j,\ \rho > 0) \quad \text{and} \quad |\lambda| = 1.$$

From the identity (2.5), we have, by Parseval's equation,

$$\begin{aligned}
\int_{R^n} |f(x+i\rho\lambda)|^2\,dx &= \int_D |F(t)|^2 e^{2\rho(\lambda,t)}\,dt \\
&\le e^{2\rho H_D(\lambda)} \int_D |F(t)|^2\,dt.
\end{aligned} \tag{4.1}$$

Hence, for $\rho\lambda \in -V^*$ we have

$$\frac{1}{2}\varlimsup_{\rho\to\infty}\frac{1}{\rho}\log\int_{R^n}|f(x+i\rho\lambda)|^2\,dx \le H_D(\lambda). \tag{4.2}$$

Further, when $f(t) \ne 0$ a.e. on D we can prove that the actual limit exists and that it is equal to $H_D(\lambda)$ as in Martin [**5**].

5. Sufficiency condition. In our situation, we will give a complete answer for our fundamental problem:

THEOREM 5.1. *$f(z)$ belongs to $H_K(T_{-V^*}, D)$ if and only if*

(5.1) *$f(z)$ is holomorphic on T_{-V^*} and $L_2(R^n, dx)$-integrable, and further, for any $y \in -V^*$, the integral*

$$\int_{R^n} |f(x+iy)|^2 \, dx$$

exists,

and

$$\frac{1}{2} \varlimsup_{\substack{\rho \to \infty \\ \rho\lambda \in -V^*}} \frac{1}{\rho} \log \int_{R^n} |f(x+i\rho\lambda)|^2 \, dx \leq H_D(\lambda). \tag{5.2}$$

PROOF. It is sufficient to prove that any function $f(z)$ satisfying (5.1) and (5.2) is an image by the Fourier-Laplace transform (2.1) of an $L_2(D, dt)$ function. Since $f(x) \in L_2(R^n, dx)$, we can define the $L_2(R^n, dt)$ function $\tilde{F}(t)$ by

$$\tilde{F}(t) = \underset{N \to \infty}{\text{l.i.m.}} \left(\frac{1}{2\pi}\right)^{n/2} \int_{|x_j|<N} f(x) e^{i(x,t)} \, dx. \tag{5.3}$$

Of course,

$$f(x) = \underset{N \to \infty}{\text{l.i.m.}} \left(\frac{1}{2\pi}\right)^{n/2} \int_{|t_j|<N} \tilde{F}(t) e^{-i(x,t)} \, dt, \tag{5.4}$$

in the framework of L_2 spaces.

We first assume that in addition $f(x) \in L_1(R^n, dx)$. Then, (5.3) exists in the ordinary sense

$$\tilde{F}(t) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{R^n} f(x) e^{i(x,t)} \, dx.$$

By condition (5.2), since the integrals $\int_{R^n} |f(x+iy)|^2 \, dx$ exist for all $y \in -V^*$, the integrals

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{R^n} f(x+iy) e^{i(x+iy,t)} \, dx \tag{5.5}$$

also exist for all $y \in -V^*$. Moreover, by using the Cauchy integral theorem, we see that the integrals (5.5) are independent of $y \in -V^*$. Hence, we set

$$\tilde{\tilde{F}}(t) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{R^n} f(x+iy) e^{i(x+iy,t)} \, dx.$$

(See [**12**, pp. 98–101] for this argument.) Then, we see immediately that $\tilde{\tilde{F}}(t)$ is continuous on R^n and $\tilde{\tilde{F}}(t) = \tilde{F}(t)$ on R^n. Hence, by Parseval's equation, we have

$$\int_{R^n} |f(x+iy)|^2 \, dx = \int_{R^n} |\tilde{F}(t)|^2 e^{2(y,t)} \, dt.$$

Hence, for $y = \rho\lambda \in -V^*$, we have

$$
(5.6) \qquad \begin{aligned} H_D(\lambda) &\geq \frac{1}{2} \overline{\lim_{\rho\to\infty}} \frac{1}{\rho} \log \int_{R^n} |f(x+i\rho\lambda)|^2 \, dx \\ &= \frac{1}{2} \overline{\lim_{\rho\to\infty}} \frac{1}{\rho} \log \int_{R^n} |\tilde{F}(t)|^2 e^{2\rho(\lambda,t)} \, dt. \end{aligned}
$$

For any $t_0 \in \overline{D}^c$, we will show that $\tilde{F}(t_0) = 0$. Since $\tilde{F}(t)$ is continuous on R^n, when $\tilde{F}(t_0) \neq 0$, for some closed sphere $\overline{S(t_0)}$ with the center t_0 ($\subset \overline{D}^c$),

$$
(5.7) \qquad |\tilde{F}(t)| \geq m > 0 \quad \text{on } \overline{S(t_0)} \text{ for some constant } m.
$$

Further then, since

$$
(5.8) \qquad (\lambda, t) > H_D(\lambda) \quad \text{on } \overline{S(t_0)},
$$

there exists $\varepsilon > 0$ such that

$$
(5.9) \qquad (\lambda, t) \geq H_D(\lambda) + \varepsilon \quad \text{on } \overline{S(t_0)}.
$$

Hence, from (5.6) we have

$$
(5.10) \qquad \begin{aligned} H_D(\lambda) &\geq \frac{1}{2} \lim_{\rho\to\infty} \frac{1}{\rho} \log \int_{\overline{S(t_0)}} |\tilde{F}(t)|^2 e^{2\rho(\lambda,t)} \, dt \\ &\geq \frac{1}{2} \lim_{\rho\to\infty} \frac{1}{\rho} \log \left\{ m^2 e^{2\rho(H_D(\lambda)+\varepsilon)} \int_{\overline{S(t_0)}} dt \right\} \\ &= H_D(\lambda) + \varepsilon, \end{aligned}
$$

which implies a contradiction. Hence, the support of $\tilde{F}$ is contained in $\overline{D}$. Further, since (2.3) is valid for any $y \in -V^*$, in (5.4) we obtain the desired expression

$$
f(z) = \left(\frac{1}{2\pi}\right)^{n/2} \int_D \tilde{F}(t) e^{-i(z,t)} \, dt.
$$

When $f(x)$ is in $L_2(R^n, dx)$ but not necessarily in $L_1(R^n, dx)$ we set

$$
f_\varepsilon(z) = f(z) \prod_{j=1}^{n} \frac{\sin \varepsilon z_j}{\varepsilon z_j}.
$$

Then we have $f_\varepsilon \in L_1(R^n, dx)$. Hence, the Fourier transform $\tilde{F}_\varepsilon(t)$ of $f_\varepsilon(z)$ vanishes outside the convex set $\overline{D}$. From the relation

$$
\tilde{F}_\varepsilon(t) = \left(\frac{1}{2\varepsilon}\right)^n \int_{t_1-\varepsilon}^{t_1+\varepsilon} \cdots \int_{t_n-\varepsilon}^{t_n+\varepsilon} \tilde{F}(\hat{t}) \, d\hat{t}
$$

we obtain

$$
\lim_{\varepsilon\to 0} \tilde{F}_\varepsilon(t) = \tilde{F}(t)
$$

for almost all $t \in R^n$. Hence, the support of $\tilde{F}$ is contained in $\overline{D}$. We thus complete the proof of the theorem.

ADDED IN PROOF. In Theorem 5.1, $f(x)$ are also considered as the boundary values such that

$$
\lim_{\substack{y\to 0 \\ y\in -V^*}} f(x+iy) = f(x)
$$

in the sense of the L_2 norm (see [**12**, Chapter III]).

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DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, GUNMA UNIVERSITY, KIRYU, 376, JAPAN

Current address: Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260