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JULIA'S LEMMA AND WOLFF'S THEOREM FOR J^*-ALGEBRAS

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ABSTRACT. Julia's lemma and Wolff's theorem are established for (Fréchet-) holomorphic maps of bounded symmetric homogeneous domains in infinite dimensional complex Banach spaces called J^*-algebras.

1. Introduction. In [**11**] V. P. Potapov extended the classical lemma of G. Julia [**8**, p. 87] to matrix-valued holomorphic maps of a complex variable. Next, I. Glicksberg [**6**] and K. Fan [**5**] proved the versions of Julia's lemma for function algebras and for holomorphic maps of proper contraction operators in the sense of functional calculus, respectively.

In another paper [**4**], K. Fan extended the classical theorem of J. Wolff [**14**] on iterates of self-maps to holomorphic maps of proper contraction operators in the sense of functional calculus. Similar extensions of Wolff's theorem to (Fréchet-) holomorphic maps of the open unit ball and the generalized upper half-plane in $\mathbf{C}^N$ were given by G. N. Chen [**1**]. Furthermore, Y. Kubota [**9**] and B. D. MacCluer [**10**], using differnet methods, proved some results on iterates of Wolff-Denjoy type [**14**, **2**] in $\mathbf{C}^N$.

The main results of this paper are of the above two types (see §2). We first prove a general version of Julia's lemma for (Fréchet-)holomorphic maps of bounded symmetric homogeneous domains in infinite dimensional complex Banach spaces called J^*-algebras. We next prove, as an application of this, the extension of Wolff's theorem. In particular, our results imply Julia's lemma and Wolff's theorem for arbitrary complex Hilbert spaces (see §5), B^*-algebras, C^*-algebras, and ternary algebras.

2. Notions. Main results. Let H and K be Hilbert spaces over $\mathbf{C}$, let $\mathcal{L}(H,K)$ denote the Banach space of all bounded linear operators X from H to K with the operator norm, and let $\mathfrak{A} \in \mathcal{L}(H,K)$ be a J^*-algebra (see L. A. Harris [**7**]), i.e. a normed complex linear subspace of $\mathcal{L}(H,K)$ closed under the operation $X \to XX^*X$.

Let

$$\mathfrak{A}_0 = \{X \in \mathfrak{A}\colon \|X\| < 1\}, \qquad \mathfrak{A}_1 = \{X \in \mathfrak{A}\colon \|X\| \leq 1\},$$

and, for $X \in \mathfrak{A}_0$, $Z \in \mathfrak{A}_1$, and $\alpha \geq 1$, let

$$W_Z(X) = (I_H - Z^*X)A_X^{-1}(I_H - X^*Z),$$
$$c_\alpha(Z,X) = \{I_H\alpha\|W_Z(X)\| + ZZ^*\}^{-1},$$

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$$r_\alpha(Z,X) = \|c_\alpha(Z,X)\|^{1/2}\, \|I_H\{\alpha\|W_Z(X)\| - 1\} + Z^* c_\alpha(Z,X) Z\|^{1/2},$$

where I_H is the identity map on H and $A_X = I_H - X^*X$.

We shall use these notations in proving the following Julia-type lemma for $\mathfrak{A}$.

LEMMA 2.1. *Let $F\colon \mathfrak{A}_0 \to \mathfrak{A}_0$ be a holomorphic map in $\mathfrak{A}_0$. If $\{z_m\} \subset \mathfrak{A}_0$ is such that*

$$\lim_{m\to\infty} \|Z_m - V\| = 0 \tag{2.1}$$

and

$$\lim_{m\to\infty} \|F(Z_m) - V\| = 0 \tag{2.2}$$

for some $V \in \mathfrak{A}_1\backslash\mathfrak{A}_0$, and if

$$\lim_{m\to\infty} \frac{\|A_{F(Z_m)}\|}{1 - \|Z_m\|^2} = \alpha \neq \infty, \tag{2.3}$$

then

$$\|X - c_1(V,X)V\| \le r_1(V,X), \tag{2.4}$$

$$\|W_V[F(X)]\| \le \alpha\|W_V(X)\|, \tag{2.5}$$

and

$$\|F(X) - c_\alpha(V,X)V\| \le r_\alpha(V,X) \tag{2.6}$$

hold for all $X \in \mathfrak{A}_0$.

We can also use Lemma 2.1 to obtain a Wolff-type theorem for $\mathfrak{A}$.

THEOREM 2.2. *Let $F\colon \mathfrak{A}_0 \to \mathfrak{A}_0$ be a compact holomorphic map having no fixed point in $\mathfrak{A}_0$ and let $F^{[n]}$ denote the nth iterate of F (i.e., $F^{[1]} = F$ and $F^{[n]} = F \circ F^{[n-1]}$ for $n \ge 2$). Then there exist $\{Z_m\} \subset \mathfrak{A}_0$ and $V \in \mathfrak{A}_1\backslash\mathfrak{A}_0$ such that*

$$\lim_{m\to\infty} \|Z_m - V\| = 0 \quad \text{and} \quad F(V) = V.$$

Moreover, $\|X - c_1(V,X)V\| \le r_1(V,X)$ for all $X \in \mathfrak{A}_0$, and if

$$\lim_{m\to\infty} \frac{\|A_{Z_m}\|}{1 - \|Z_m\|^2} = \alpha \neq \infty, \tag{2.7}$$

then

$$\|W_V[F^{[n]}(X)]\| \le \alpha\|W_V(X)\| \tag{2.8}$$

and

$$\|F^{[n]}(X) - c_\alpha(V,X)V\| \le r_\alpha(V,X) \tag{2.9}$$

hold for all $x \in \mathfrak{A}_0$ and $n = 1, 2, \ldots$.

For the special case when $\mathfrak{A} = H$, see §5.

3. Proof of Lemma 2.1.

LEMMA 3.1. *If $F\colon \mathfrak{A}_0 \to \mathfrak{A}_0$ is a holomorphic map in $\mathfrak{A}_0$, then*

$$\|W_{F(Z)}[F(X)]\| \leq \frac{\|A_{F(Z)}\|}{1-\|Z\|^2}\|W_Z(X)\| \tag{3.1}$$

for all $X, Z \in \mathfrak{A}_0$.

PROOF. Let $X, Z \in \mathfrak{A}_0$. It follows from [**13**, Theorem 1(a)] that

$$\|A_{F(Z)}^{-1/2} W_{F(Z)}[F(X)] A_{F(Z)}^{-1/2}\| \leq \|A_Z^{-1/2} W_Z(X) A_Z^{-1/2}\|. \tag{3.2}$$

But

$$\|A_Z^{-1/2}\|^2 = \|A_Z^{-1}\| = (1-\|Z\|^2)^{-1} \tag{3.3}$$

and

$$I_H\|A_{F(Z)}^{1/2}\|^{-1} \leq A_{F(Z)}^{-1/2} \quad \text{since } I_H \leq A_{F(Z)}^{-1/2}. \tag{3.4}$$

Thus, using (3.3) and (3.4), from (3.2) we get (3.1). This completes the proof.

LEMMA 3.2. *If $X \in \mathfrak{A}_0$, $Z \in \mathfrak{A}_1$, and D satisfy*

$$\|W_Z(X)\| \leq D, \tag{3.5}$$

then

$$\begin{aligned} &\|X-(I_H D + ZZ^*)^{-1}Z\| \\ &\quad \leq \|(I_H D + ZZ^*)^{-1}\|^{1/2}\|I_H(D-1) + Z^*(I_H D + ZZ^*)^{-1}Z\|^{1/2}. \end{aligned} \tag{3.6}$$

PROOF. Inequality (3.5) implies

$$\|A_X^{-1/2}(I_H - X^*Z)(I_H - Z^*X)A_X^{-1/2}\| \leq D$$

or, equivalently,

$$(I_H - X^*Z)(I_H - Z^*X) \leq A_X D.$$

Further, the above inequality is identical to

$$\begin{aligned} &\{X^* - Z^*(I_H D + ZZ^*)^{-1}\}(I_H D + ZZ^*)\{X - (I_H D + ZZ^*)^{-1}Z\} \\ &\quad \leq I_H(D-1) + Z^*(I_H D + ZZ^*)^{-1}Z. \end{aligned}$$

Consequently,

$$\|(I_H D + ZZ^*)^{1/2}\{X-(I_H D+ZZ^*)^{-1}Z\}\|^2 \leq \|I_H(D-1)+Z^*(I_H D+ZZ^*)^{-1}Z\}.$$

Hence (3.6) follows. This completes the proof.

Now, we assume that $\{Z_m\} \subset \mathfrak{A}_0$ satisfies (2.1)–(2.3). Then, from Lemma 3.1 it follows that (2.5) holds, and (2.4) and (2.6) follow from the relations $\|W_Z(X)\| = D_1$ and $\|W_V[F(X)]\| \leq D_\alpha$, $D_\alpha = \alpha\|W_V(X)\|$, respectively, if we use Lemma 3.2.

4. Proof of Theorem 2.2.

LEMMA 4.1. *If $F: \mathfrak{A}_0 \to \mathfrak{A}_0$ is a compact holomorphic map having no fixed point in $\mathfrak{A}_0$, then there exist $\{Z_m\} \subset \mathfrak{A}_0$ and a fixed point $V \in \mathfrak{A}_1 \backslash \mathfrak{A}_0$ of F such that*

$$\lim_{m\to\infty} \|Z_m - V\| = 0.$$

PROOF. Let $F_m = \alpha_m F$, $0 < \alpha_m < 1$, $m = 1, 2, \ldots$, and let $\lim_{m\to\infty} \alpha_m = 1$. Let $\{Z_m\} \subset \mathfrak{A}_0$ be such that $F_m(Z_m) = Z_m$, $m = 1, 2, \ldots$; such a sequence exists by the Earle-Hamilton theorem [**3**]. Since $F(\mathfrak{A}_0)$ is contained in a compact subset of $\mathfrak{A}$, F has no fixed points in $\mathfrak{A}_0$ and $F(Z_m) = Z_m/\alpha_m$, $m = 1, 2, \ldots$, we may assume that $\lim_{m\to\infty} \|Z_m - V\| = 0$ and $\lim_{m\to\infty} \|F(Z_m) - V\| = \lim_{m\to\infty} \|Z_m/\alpha_m - V\| = 0$ for some $V \in \mathfrak{A}_1 \backslash \mathfrak{A}_0$. This completes the proof.

Let now $X \in \mathfrak{A}_0$ and let $F_m^{[n]}$ denote the nth iterate of F_m, $m = 1, 2, \ldots$. By Lemma 4.1 and (3.1),

$$\|W_{Z_m}[F_m^{[n]}(X)\| \leq \frac{\|A_{Z_m}\|}{1 - \|Z_m\|^2} \|W_{Z_m}(X)\|. \tag{4.1}$$

Furthermore (see [**12**, formula (4.6), p. 158]),

$$\lim_{m\to\infty} \|F_m^{[n]}(X) - F^{[n]}(X)\| = 0. \tag{4.2}$$

Thus, if (2.7) holds, from (4.1) and (4.2) we obtain (2.8). By Lemma 3.2, inequality (2.8) implies (2.9).

5. Concluding remarks. Let

$$H_0 = \{x \in H: \|x\| < 1\}, \qquad H_1 = \{x \in H: \|x\| \leq 1\},$$

and, for $x \in H_0, v \in H_1 \backslash H_0$, and $\alpha \geq 1$, let

$$c_\alpha(v, x) = \frac{1 - \|x\|^2}{\alpha|1 - \langle x, v\rangle|^2 + 1 - \|x\|^2},$$
$$r_\alpha(v, x) = \frac{\alpha|1 - \langle x, v\rangle|^2}{\alpha|1 - \langle x, v\rangle|^2 + 1 - \|x\|^2}.$$

Identifying H with $\mathcal{L}(\mathbf{C}, H)$, as corollaries from our main results and their proofs we get the following Julia's lemma and Wolff's theorem for arbitrary complex Hilbert spaces.

LEMMA 5.1. *Let $F: H_0 \to H_0$ be a holomorphic map in H_0. If $\{z_m\} \subset H_0$ is such that*

$$\lim_{m\to\infty} \|z_m - v\| = 0 \quad \textit{and} \quad \lim_{m\to\infty} \|F(z_m) - v\| = 0$$

for some $v \in H_1 \backslash H_0$, and if

$$\lim_{m\to\infty} \frac{1 - \|F(z_m)\|^2}{1 - \|z_m\|^2} = \alpha \neq \infty,$$

then

$$\|x - c_1(v, x)v\| = r_1(v, x),$$
$$\frac{|1 - \langle F(x), v\rangle|^2}{1 - \|F(x)\|^2} \leq \alpha \frac{|1 - \langle x, v\rangle|^2}{1 - \|x\|^2},$$

and

$$\|F(x) - c_\alpha(v,x)v\| \le r_\alpha(v,x)$$

for all $x \in H_0$.

THEOREM 5.2. *Let* $F\colon H_0 \to H_0$ *be a compact holomorphic map having no fixed points in* H_0 *and let* $F^{[n]}$ *denote the nth iterate of* F. *Then there exists* $v \in H_1 \backslash H_0$ *such that* $F(v) = v$,

$$\|x - c_1(v,x)v\| = r_1(v,x),$$

$$\frac{|1 - \langle F^{[n]}(x), v\rangle|^2}{1 - \|F^{[n]}(x)\|^2} \le \frac{|1 - \langle x, v\rangle|^2}{1 - \|x\|^2},$$

and

$$\|F^{[n]}(x) - c_1(v,x)v\| \le r_1(v,x)$$

for all $x \in H_0$ *and* $n = 1, 2, \ldots$.

REMARKS. If $H = \mathbf{C}$, Lemma 5.1 and Theorem 5.2 imply the results of Julia [**8**, p. 87] and Wolff [**14**], respectively.

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