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UNBOUNDED COMPOSITION OPERATORS ON $H^2(B_2)$

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ABSTRACT. Examples are given of holomorphic self-maps of the unit ball on $\mathbf{C}^2$ which induce unbounded composition operators on the Hardy space H^2. In particular, an example is given which is one-to-one on the closed ball. Also, a valence condition on the boundary of this ball is given which is sufficient for unboundedness of the induced composition operator.

1. Introduction. Let B_n be the open unit ball in $\mathbf{C}^n$ and let $H^2 = H^2(B_n)$ be the Hardy space on B_n. If ϕ is a holomorphic mapping of B_n into B_n, then the composition operator $C_\phi\colon f \to f \circ \phi$ maps holomorphic functions on B_n into holomorphic functions. If $n = 1$, it is well known that C_ϕ is a bounded operator on H^2 (see [**5**], e.g.). For $n > 1$, there are many examples (see [**1**, **2**]) which show that C_ϕ need not be bounded. These examples exhibit a "collapsing" property on the boundary ∂B_n of B_n. For instance ϕ may map an arc on ∂B_n to a point on ∂B_n. The main result of this note is the construction (Theorem 2) of a mapping $\Phi\colon \overline{B}_2 \to \overline{B}_2$ which is holomorphic and one-to-one on $\overline{B}_2$ and such that C_Φ is unbounded on H^2. Φ is in fact a polynomial mapping.

B. MacCluer and J. Shapiro show in [**4**, Theorem 6.4] that if $\phi\colon B_n \to B_n$ is one-to-one and if the derivative of ϕ^{-1} is bounded on $\phi(B_n)$, then C_ϕ is bounded on H^2 (see also [**1**, Theorem 2]). Our example shows that even for one-to-one mappings, some additional hypothesis on ϕ must be imposed to guarantee that C_ϕ is bounded. Example 4 is also related to the above theorem. In Theorem 1 we give a valence condition on ϕ which is sufficient for unboundedness of C_ϕ. All of our results rely on the following Carleson measure criterion for boundedness of C_ϕ.

THEOREM [**3**]. *Suppose that $\phi\colon B \to B$ is holomorphic and that $\mu = \sigma(\phi^*)^{-1}$. Then C_ϕ is bounded on H^2 if and only if there is a $C > 0$ so that $\mu(S(\varsigma, t)) \leq Ct^2$ for all $\varsigma \in \partial B$ and $t > 0$. In this case we say that μ is a σ-Carleson measure.*

W. Rudin's book [**6**] will be used as a standard reference. We will restrict our attention to $B_2 = B$. For $\phi\colon B \to B$, write $\phi = (\phi_1, \phi_2)$. Let σ denote surface measure on ∂B. If $\varsigma \in \partial B$, set $\phi^*(\varsigma) = \lim_{r\to 1} \phi(r\varsigma)$; so $\phi^*\colon \partial B \to \overline{B}$. Further define $S(\varsigma, t) = \{z \in \overline{B}\colon |1 - \langle z, \varsigma\rangle| < t\}$. Here $\langle\cdot,\cdot\rangle$ denotes the usual complex inner product in $\mathbf{C}^2$, and $t > 0$. Let $Q(\varsigma, t) = S(\varsigma, t) \cap \partial B$.

2. A criterion for unboundedness. In this section we prove the following.

THEOREM 1. *Suppose that $\phi\colon B \to B$ is holomorphic on B and that ϕ' is uniformly bounded on B. If $\sup\{\operatorname{card}(\phi^*)^{-1}(\xi)\colon \xi \in \partial B\} = \infty$, then C_ϕ is unbounded on H^2.*

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The proof of this theorem depends on the following lemma. We assume the smoothness hypothesis of Theorem 1.

LEMMA 1. *Suppose that* $\phi(0) = 0$. *Then there exist positive numbers* A *and* δ *which satisfy the following. If* $\varsigma, \xi \in \partial B$ *and* $\phi^*(\varsigma) = \xi$, *then* $\phi^*(Q(\varsigma, t)) \subset S(\varsigma, At)$ *for all* $0 < t < \delta$.

PROOF. ϕ has a continuous extension to $\overline{B}$, which we can also denote by ϕ. In fact ϕ is Lipschitz on $\overline{B}$. Thus there is a $D > 0$ so that if $z, w \in \overline{B}$ and $|z - w| < t$, then $|\phi(z) - \phi(w)| < Dt$. Let $e = (1, 0)$. Consider the case that $\varsigma = \xi = e$. Set

$$L = \liminf_{z \to e} \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

Then by the Julia-Carathéodory theory [**6**, pp. 174–181], $L = \lim_{r \to 1} D_1 \phi_1(re)$. Note that $L \geq 1$ by the Schwarz Lemma. For $0 < c < 1$, consider the ellipsoids

$$E_c = \left\{ z \in B \colon \frac{|z_1 - (1 - c)|^2}{c^2} + \frac{|z_2|^2}{c} < 1 \right\}.$$

By [**6**, Theorem 8.54], we have $\phi(E_c) \subset E_{Lc}$ if $c < 1/L$. Also note that $E_c \subset S(e, 2c)$.

Now if $z \in Q(e, t)$, we have $|1 - z_1| < t$, so that $|z_1| > 1 - t$. Hence $|z_2|^2 = 1 - |z_1|^2 < 2t - t^2$. If follows that $(1 - 2t, z_2) \in E_{2t}$. Thus $(1 - 2t, z_2) \in S(e, 4t)$, so that $\phi(1 - 2t, z_2) \in S(e, 4tL)$.

Set $\delta = 1/2L$. Suppose that $0 < t < \delta$, and $z \in Q(e, t)$. Then $|z - (1 - 2t, z_2)| \leq |1 - z_1| + 2t < 3t$, so that $|\phi(z) - \phi(1 - 2t, z_2)| < D(3t)$. Thus $|1 - \phi_1(z)| < 4tL + 3tD$, and the lemma holds with $A = 4L + 3D$.

For the general case choose unitaries U and $V \colon \mathbf{C}^2 \to C^2$ with $Ue = \varsigma$ and $V\xi = e$. Apply the first part of the proof to the map $\lambda = V \circ \phi \circ U$. There are positive numbers A and δ so that $\lambda(Q(e, t)) \subset S(e, At)$ for $0 < t < \delta$. Since $U(Q(e, t)) = Q(\varsigma, t)$ and $V^{-1}(S(e, At)) = S(\xi, At)$, we have $\phi(Q(\varsigma, t)) \subset S(\xi, At)$.

Finally, note that A depends on the Lipschitz constant D and on L. But $L \leq \sup\{\|\phi'(z)\| \colon z \in B\}$, so that both δ and A can be chosen independent of ς and ξ.

PROOF OF THEOREM 1. Since an automorphism of B induces a bounded composition operator, we may assume that $\phi(0) = 0$. Fix a positive integer n. Suppose that $\xi \in \partial B$ and $\operatorname{card}(\phi^*)^{-1}(\xi) \geq n$. Choose $\varsigma_1, \varsigma_2, \ldots, \varsigma_n \in \partial B$ so that $\phi^*(\varsigma_k) = \xi$, $1 \leq k \leq n$. Choose A and δ as in Lemma 1. Then choose t_0 with $0 < t_0 \leq \delta$ and so that if $0 < t < t_0$, the sets $Q(\varsigma_1, t), \ldots, Q(\varsigma_n, t)$ are pairwise disjoint. Thus

$$\sigma(\phi^*)^{-1}(S(\xi, At)) \geq \sigma\left(\bigcup_1^n Q(\varsigma_k, t)\right) \approx nt^2.$$

Since n is arbitrary, it is clear that $\sigma(\phi^*)^{-1}$ is not a Carleson measure, and the theorem is proven.

3. Examples.

EXAMPLE 1. This example is a slight variant of an example shown to us by J. P. Rosay. Let

$$\psi(z_1, z_2) = \tfrac{1}{2}(1 + z_1^2 + z_2^2, z_2(1 - z_1^2 - z_2^2)).$$

If $z \in \overline{B}$, then

$$\begin{aligned}|\psi(z)|^2 &= \tfrac{1}{4}(1 + 2\operatorname{Re}(z_1^2 + z_2^2) + |z_1^2 + z_2^2|^2 + |z_1|^2(1 - 2\operatorname{Re}(z_1^2 + z_2^2) + |z_1^2 + z_2^2|^2)) \\ &\le \tfrac{1}{4}(2 + 2|z_1^2 + z_2^2|^2) \le 1.\end{aligned}$$

Further, $|\psi(z)| = 1$ if and only if $z_1^2 + z_2^2 = 1$, in which case $\psi(z) = e$. Thus $(\psi^*)^{-1}(e)$ is the unit circle C in the $\operatorname{Re} z_1, \operatorname{Re} z_2$ plane. Hence C_ψ is unbounded on H^2, by Theorem 1. It can be shown directly that $\sigma(\psi^*)^{-1}(S(e,t)) \approx t^{3/2}$. We observe some additional properties of ψ. Consider the complex Jacobian $J\psi$ on $\overline{B}$. It is easy to check that $J\psi$ vanishes only on C and on the complex line $z_1 = 0$. Also if z and w are in $\overline{B} - C$ and $\psi(z) = \psi(w)$, we have $z_1 = \pm w_1$ and $z_2 = w_2$. Thus ψ is a two-to-one map on $\overline{B} - C$. ψ is one-to-one on $\{z \in B\colon \operatorname{Re} z_1 > 0\}$.

EXAMPLE 2. Let $\rho(z_1, z_2) = (1 - \sqrt{\tfrac{1}{2}(1 - z_1^2 - z_2^2)}, \tfrac{1}{2}z_2(1 - z_1^2 - z_2^2))$. Here $\sqrt{}$ denotes the principal branch of the square root. Then ρ shares may properties with ψ. An application of the Schwarz Lemma shows that $|\rho_1(z)| \le |\psi_1(z)|$ for $z \in B$. It follows that $\rho(\overline{B} - C) \subset B$. Also $\rho(c) = \{e\}$, and ρ is two-to-one on $\overline{B} - C$. ρ is continuous on $\overline{B}$, but ρ' is not bounded on B.

We will show that C_ρ is compact on H^2. First we show that $\rho(B)$ is contained in a Koranyi approach region $D_\alpha(e) = \{z \in B\colon |1 - z_1| < (\alpha/2)(1 - |z|^2)\}$.

Let $E_1 = \{z \in \overline{B}\colon |1 - z_1^2 - z_2^2| < \tfrac{1}{4}\}$ and $E_2 = \overline{B} - E_1$. Then $\sup\{|\rho(z)|\colon z \in E_2\} < 1$, so we have $\rho(E_2) \subset D_{\alpha_0}(e)$ for some $\alpha_0 > 0$. If $z \in E_1$, then

$$|\rho(z)|^2 \le 1 - 2\operatorname{Re}\sqrt{\tfrac{1}{2}(1 - z_1^2 - z_2^2)} + \tfrac{1}{2}|1 - z_1^2 - z_2^2| + \tfrac{1}{4}|1 - z_1^2 - z_2^2|^2.$$

Thus

$$1 - |\rho(z)|^2 \ge 2\operatorname{Re}\sqrt{\tfrac{1}{2}(1 - z_1^2 - z_2^2)} - |1 - z_1^2 - z_2^2|.$$

But $\operatorname{Re}(1 - z_1^2 - x_2^2) \ge 0$, so

$$\operatorname{Re}\sqrt{1 - z_1^2 - z_2^2} \ge \frac{1}{\sqrt{2}}|1 - z_1^2 - z_2^2|^{1/2}.$$

Hence

$$\begin{aligned}1 - |\rho(z)|^2 &\ge |1 - z_1^2 - z_2^2|^{1/2}(1 - |1 - z_1^2 - z_2^2|^{1/2}) \\ &\ge \tfrac{1}{2}|1 - z_1^2 - z_2^2|^{1/2} = \tfrac{1}{2}|1 - \rho_1(z)|.\end{aligned}$$

So $\rho(B) \subset D_\alpha(e)$, where $\alpha = \max(\alpha_0, 4)$.

By the computation mentioned in Example 1, we have

$$\sigma(\rho^*)^{-1}(S(e,t)) = \sigma(\psi^*)^{-1}(S(e, 2t^2)) \approx t^3.$$

By [**3**, Lemma 2.1, (ii)], C_ρ is compact.

We now construct a biholomorphism Φ of B into B which is a homeomorphism of $\overline{B}$ onto $\Phi(\overline{B})$ and such that C_Φ is unbounded on H^2. Let ψ be as in Example 1 and let

$$\phi(z) = \tfrac{1}{25}(18 + 9z_1 - 2z_1^2 + 2z_2^2, 9z_2 - 4z_1z_2).$$

We will consider the map $\Phi = \psi \circ \phi$. Our first step is to study ϕ.

LEMMA 2. *Let $f(z) = 18+9z-2z^2$. If $|z| \le r < 1$ and $z \ne r$, then $|f(z)| < f(r)$.*

PROOF. Let $z = re^{i\theta} = x + iy$, $0 < r < 1$. Then

$$\begin{aligned}|f(z)|^2 &= 18 + 9^2r^2 + 2^2r^4 + 2\cdot 9\cdot 18x - 2\cdot 2\cdot 18(x^2-y^2) - 2\cdot 2\cdot 9r^2x\\ &= 468 + 153r^2 + 4r^4 - 144(1-x)^2 + 36x(1-r^2)\\ &\le 468 + 153r^2 + 4r^4 - 144(1-r)^2 + 36r(1-r^2) = f(r)^2,\end{aligned}$$

with equality if and only if $x = r$.

NOTE. $g(z) = f(z)/25$ is the second Taylor polynomial at $z = 1$ of the automorphism $A(z) = (z + 2/3)(1 + 2z/3)^{-1}$ of the unit disc Δ. Using Lemma 2, one can see that $g(\Delta) \subset \Delta$. Also g is univalent on $\overline{\Delta}$, $g(1) = 1$, and the range of g has second order contact at 1 with the unit circle.

LEMMA 3. *ϕ is one-to-one on $\overline{B}$.*

PROOF. Suppose that $\phi(z) = \phi(w)$ with $z, w \in \overline{B}$. Then $9z_1 - 2z_1^2 + 2z_2^2 = 9w_1 - 2w_1^2 + 2w_2^2$, so that $(z_1 - w_1)(9 - 2z_1 - 2w_1) = 2(w_2 - z_2)(w_2 + z_2)$. Hence $|z_1 - w_1| \le \frac{4}{5}|w_2 - z_2|$.

Also $9z_2 - 4z_1z_2 = 9w_2 - 4w_1w_2$ so that $(9 - 4z_1)(z_2 - w_2) = 4w_2(z_1 - w_1)$. Thus $|z_2 - w_2| \le \frac{4}{5}|z_1 - w_1|$, and $z = w$.

LEMMA 4. *$\phi(e) = e$, and $\phi(\overline{B} - \{e\}) \subset B$.*

PROOF. For $z \in \partial B$, write $z_1 = re^{i\theta} = x + iy$. Then $|z_2| = \sqrt{1-r^2}$. Lemma 2 is used in the following inequality.

$$\begin{aligned}25|\phi(z)|^2 &\le (|18 + 9z_1 - 2z_1^2| + 2|z_2|^2)^2 + |z_2|^2|9 - 4z_1|^2\\ &\le |f(z_1)|^2 + 4f(r)(1-r^2) + 4(1-r^2)^2 + (1-r^2)(81 - 72x + 16r^2)\\ &= 468 + 153r^2 + 4r^4 - 144(1-x)^2 + 36x(1-r^2)\\ &\quad + 4(18 + 9r - 2r^2)(1-r^2)\\ &\quad + 4(1-r^2)^2 + (1-r^2)(81 - 72x + 16r^2)\\ &= 625 - 144(1-x)^2 + 36(1+r)(1-r)(r-x) \le 625\end{aligned}$$

since $36(1+r)(1-r)(r-x) \le 72(1-x)^2 \le 144(1-x)^2$. Also note that equality holds if and only if $r = x = 1$.

The motivation behind the formula for ϕ is that if z_1 and z_2 are real, then $\phi_1(z_1, z_2) = \operatorname{Re} g(z_1 + iz_2)$ and $\phi_2(z_1, z_2) = \operatorname{Im} g(z_1 + iz_2)$. Since $g(\Delta) \subset \Delta$, one can hope that $\phi(B) \subset B$. Further, the curve $\phi(C)$ has second order tangency at e to C.

THEOREM 2. *$\Phi(\overline{B}) \subset \overline{B}$, Φ is a homeomorphism of B onto $\Phi(\overline{B})$, and C_Φ is unbounded on H^2.*

PROOF. Since ϕ and ψ both map $\overline{B}$ into $\overline{B}$, we have $\Phi(\overline{B}) \subset \overline{B}$. Also $\operatorname{Re}\phi_1(f) > 0$ for $z \in \overline{B}$ and ψ is one-to-one on $B \cap \{z\colon \operatorname{Re} z_1 > 0\}$. Hence Lemmas 3 and 4 show that Φ is a homeomorphism. It remains to show that C_Φ is unbounded. Now

$$\Phi_1(z) = \frac{1}{2}\left[1 + \frac{1}{25^2}((18 + 9z_1 - 2z_1^2 + 2z_2^2)^2 + (9z_2 - 4z_1z_2)^2)\right],$$

and a computation shows that

$$1-\Phi_1(z)=\frac{1}{1250}[144(1-z_1)^2+(1-z_1^2-z_2^2)(157-36z_1+4z_1^2+4z_2^2)].$$

Thus,

$$(1)\qquad |1-\Phi_1(z)|\le \tfrac{1}{6}(|1-z_1|^2+|1-z_1^2-z_2^2|)\quad \text{for } z\in\overline{B}.$$

We will show that $\lim_{t>0}(\sigma(\Phi^*)^{-1}(S(e,t))/t^2)=\infty$ so that $\sigma(\Phi^*)^{-1}$ is not a Carleson measure.

Consider the parametrization of ∂B given by $(z_1,z_2)=(\sqrt{1-\rho}e^{i\theta_1},\sqrt{\rho}e^{i\theta_2})$; $0\le\rho\le1$, $-\pi<\theta_1,\theta_2\le\pi$. It is easy to check that $d\sigma=d\rho d\theta_1 d\theta_2$. For $0<t<1$, let

$$B_t=\{(\sqrt{1-\rho}e^{i\theta_1},\sqrt{\rho}e^{i\theta_2})\colon 0<\theta_1<t,\ 0<\theta_2<\pi,\rho<\min\{\sqrt{t},(t/\theta_2)\}.$$

The following estimates show that $B_t\subset(\Phi^*)^{-1}(S(e,t))$. Suppose that $z\in B_t$. Then

$$(2)\qquad \begin{aligned}|1-z_1|^2&=1+|z_1|^2-2|z_1|\cos\theta_1\le(1-|z_1|)^2+\theta_1^2\\ &=(1-\sqrt{1-\rho})^2+\theta_1^2<\rho^2+\theta_1^2<t+t^2<2t.\end{aligned}$$

Also

(3)
$$\begin{aligned}|1-z_1^2-z_2^2|&=|1-(1-\rho)e^{2i\theta_1}-\rho e^{2i\theta_2}|\le\rho|1-e^{2i\theta_2}|+(1-\rho)|1-e^{2i\theta_1}|\\ &\le 2\rho\theta_2+2\theta_1<2t+2t=4t.\end{aligned}$$

Thus from (1), (2), and (3), $|1-\Phi_1(z)|<\frac{1}{6}(2t+4t)=t$.
Finally,

$$\begin{aligned}\sigma(B_t)&=\int_0^t d\theta_1\left[\int_0^{\sqrt{t}}d\theta_2\int_0^{\sqrt{t}}d\rho+\int_{\sqrt{t}}^{\pi}d\theta_2\int_0^{t/\theta_2}d\rho\right]=t^2+t\int_{\sqrt{t}}^{\pi}\frac{t}{\theta_2}d\theta_2\\ &=t^2+t^2\ln\pi+t^2\ln\frac{1}{\sqrt{t}}\ge\frac{t^2}{2}\ln\frac{1}{t}.\end{aligned}$$

Thus $\sigma(\Phi^*)^{-1}$ is not a Carleson measure.

Φ is the simplest one-to-one map we have been able to construct which induces an unbounded composition operator. However, motivated by inequalities (2) and (3), we can construct a simple (quadratic) mapping Λ of B into B which is two-to-one on $\overline{B}$ and so that C_Λ is unbounded.

EXAMPLE 3. Consider $\Lambda(z)=\frac{1}{9}(5+5z_1-z_1^2+\frac{3}{2}z_2^2,z_2^2)$. Just as in Lemma 2, one can show that

$$|5+5z_1-z_1^2|\le5+5r-r^2\quad\text{if } |z_1|=r<1.$$

Thus if $z\in\partial B$ and $|z_1|=r$, we have

$$\begin{aligned}|\Lambda(z)|^2&\le|\Lambda_1(z)|+|\Lambda_2(z)|^2\le\tfrac{1}{9}\left(|5+5z_1-z_1^2|+\tfrac{3}{2}|z_2|^2\right)+\tfrac{1}{81}(1-r^2)^2\\ &\le\tfrac{1}{9}[5+5r-r^2+\tfrac{3}{2}(1-r^2)+\tfrac{1}{9}(1+r)^2(1-r)^2]\\ &\le\tfrac{1}{9}\left[9-\tfrac{5}{2}(1-r)^2+\tfrac{4}{9}(1-r)^2\right]\le1,\end{aligned}$$

with equality only if $z_1 = r = 1$. Thus $\Lambda(\overline{B} - \{e\}) \subset B$. It is elementary to check that $\Lambda(z) = \Lambda(w)$ if ond only if $z_2^2 = w_2^2$, so that Λ is two-to-one on $\overline{B}$.

$$\begin{aligned} 1 - \Lambda_1(z) &= 1 - \tfrac{1}{9}\left(5 + 5z_1 - z_1^2 + \tfrac{3}{2}z_2^2\right) \\ &= \tfrac{1}{18}[5(1 - z_1)^2 + 3(1 - z_1^2 - z_2^2)], \end{aligned}$$

so the same argument as in the end of the proof of Theorem 2 shows that $\sigma(\Lambda^*)^{-1}$ is not a Carleson measure. We close with the following example.

EXAMPLE 4. Let $\phi(z) = \frac{1}{2}(1 + z_1, z_2)$ and let ψ be as in Example 1. Set $\Phi = \psi \circ \phi$. Then just as for the map of Theorem 2, we have that Φ is a homeomorphism of $\overline{B}$ onto $\Phi(\overline{B})$, $\Phi(e) = e$, and $\Phi(\overline{B} - \{e\}) \subset B$. Also the derivative of Φ^{-1} is unbounded near e. We claim that C_Φ is bounded, even though the MacCluer-Shapiro Theorem [**4**, Theorem 6.4] does not apply. The proof is somewhat tedious, and we only give an outline. We must show that there is a $C > 0$ so that if $\varsigma \in \partial B$ and $t > 0$, then $\sigma(\Phi^*)^{-1}(S(\varsigma, t)) \le Ct^2$. Since on the complement of a neighborhood of e, $|\Phi^*|$ is strictly less than 1, we need only consider ς near 1. Then if $z \in Q(\varsigma, t)$ and t is small, we will also have z near e.
If $|1 - \langle \Phi(z), \varsigma \rangle| < t$, then

$$\frac{|\varsigma_1|}{8}\left|\frac{8}{\varsigma_1} - 4 - (1 + z_1)^2 - z_2^2\right| - \frac{|\varsigma_2|\,|z_2|}{16}|4 - (1 + z_1)^2 - z_2^2| < t. \tag{4}$$

But

$$\begin{aligned} |4 - (1 + z_1)^2 - z_2^2| &\le |1 - z_1|\,|3 + z_1| + (1 - |z_1|^2) \\ &\le 4|1 - z_1| + 2|1 - z_1| = 6|1 - z_1|. \end{aligned} \tag{5}$$

Let $\lambda = 2\sqrt{2/\varsigma_1 - 1} - 1$. Then $|\lambda| > 1$ and λ is near 1.

$$\begin{aligned} |8/\varsigma_1 - 4 - (1 + z_1)^2 - z_2^2| &\ge |\lambda - z_1|\,|\lambda + 2 + z_1| - |z_2|^2 \\ &\ge 3|\lambda - z_1| - 2(1 - |z_1|) \ge |\lambda - z_1|. \end{aligned} \tag{6}$$

From (4), (5), and (6) we have

$$\frac{|\varsigma_1|}{8}|\lambda - z_1| - \frac{\sqrt{1 - |\varsigma_1|^2}\sqrt{1 - |z_1|^2}}{16} 6|1 - z_1| < t. \tag{7}$$

Some computation shows that

$$\sqrt{1 - |\varsigma_1|^2}\sqrt{1 - |z_1|^2} \le 2\left(2\sqrt{2 - |\varsigma_1|} - 1 - |z_1|\right) \le 2|\lambda - z_1|.$$

Hence if ς and z are sufficiently near e that $|\varsigma_1|/8 > \frac{1}{10}$ and $|1 - z_1| < \frac{1}{20}$, then from (7),

$$\tfrac{1}{10}|\lambda - z_1| - \tfrac{1}{20}|\lambda - z_1| < t.$$

Thus $z \in Q((\lambda/|\lambda|, 0), 20t)$. We can take $C = 400$.

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