UNBOUNDED COMPOSITION OPERATORS ON $H^2(B_2)$

J. A. CIMA AND W. R. WOGEN

ABSTRACT. Examples are given of holomorphic self-maps of the unit ball on $\mathbb{C}^2$ which induce unbounded composition operators on the Hardy space $H^2$. In particular, an example is given which is one-to-one on the closed ball. Also, a valence condition on the boundary of this ball is given which is sufficient for unboundedness of the induced composition operator.

1. Introduction. Let $B_n$ be the open unit ball in $\mathbb{C}^n$ and let $H^2 = H^2(B_n)$ be the Hardy space on $B_n$. If $\phi$ is a holomorphic mapping of $B_n$ into $B_n$, then the composition operator $C_{\phi} : f \to f \circ \phi$ maps holomorphic functions on $B_n$ into holomorphic functions. If $n = 1$, it is well known that $C_{\phi}$ is a bounded operator on $H^2$ (see [5], e.g.). For $n > 1$, there are many examples (see [1, 2]) which show that $C_{\phi}$ need not be bounded. These examples exhibit a "collapsing" property on the boundary $\partial B_n$ of $B_n$. For instance $\phi$ may map an arc on $\partial B_n$ to a point on $\partial B_n$. The main result of this note is the construction (Theorem 2) of a mapping $\phi : B_2 \to B_2$ which is holomorphic and one-to-one on $B_2$ and such that $C_{\phi}$ is unbounded on $H^2$. $\phi$ is in fact a polynomial mapping.

B. MacCluer and J. Shapiro show in [4, Theorem 6.4] that if $\phi : B_n \to B_n$ is one-to-one and if the derivative of $\phi^{-1}$ is bounded on $\phi(B_n)$, then $C_{\phi}$ is bounded on $H^2$ (see also [1, Theorem 2]). Our example shows that even for one-to-one mappings, some additional hypothesis on $\phi$ must be imposed to guarantee that $C_{\phi}$ is bounded. Example 4 is also related to the above theorem. In Theorem 1 we give a valence condition on $\phi$ which is sufficient for unboundedness of $C_{\phi}$. All of our results rely on the following Carleson measure criterion for boundedness of $C_{\phi}$.

THEOREM [3]. Suppose that $\phi : B \to B$ is holomorphic and that $\mu = \sigma(\phi^*)^{-1}$. Then $C_{\phi}$ is bounded on $H^2$ if and only if there is a $C > 0$ so that $\mu(S(\zeta, t)) \leq Ct^2$ for all $\zeta \in \partial B$ and $t > 0$. In this case we say that $\mu$ is a $\sigma$-Carleson measure.

W. Rudin's book [6] will be used as a standard reference. We will restrict our attention to $B_2 = B$. For $\phi : B \to B$, write $\phi = (\phi_1, \phi_2)$. Let $\sigma$ denote surface measure on $\partial B$. If $\zeta \in \partial B$, set $\phi^*(\zeta) = \lim_{r \to 1} \phi(r\zeta)$; so $\phi^* : \partial B \to \overline{B}$. Further define $S(\zeta, t) = \{z \in \overline{B} : |1 - \langle z, \zeta \rangle| < t\}$. Here $\langle \cdot, \cdot \rangle$ denotes the usual complex inner product in $\mathbb{C}^2$, and $t > 0$. Let $Q(\zeta, t) = S(\zeta, t) \cap \partial B$.

2. A criterion for unboundedness. In this section we prove the following.

THEOREM 1. Suppose that $\phi : B \to B$ is holomorphic on $B$ and that $\phi'$ is uniformly bounded on $B$. If $\sup\{\text{card}(\phi^*)^{-1}(\xi) : \xi \in \partial B\} = \infty$, then $C_{\phi}$ is unbounded on $H^2$.______

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The proof of this theorem depends on the following lemma. We assume the smoothness hypothesis of Theorem 1.

**Lemma 1.** Suppose that \( \phi(0) = 0 \). Then there exist positive numbers \( A \) and \( \delta \) which satisfy the following. If \( \zeta, \xi \in \partial B \) and \( \phi^*(\zeta) = \xi \), then \( \phi^*(Q(\zeta, t)) \subset S(\zeta, At) \) for all \( 0 < t < \delta \).

**Proof.** \( \phi \) has a continuous extension to \( \overline{B} \), which we can also denote by \( \phi \). In fact \( \phi \) is Lipschitz on \( \overline{B} \). Thus there is a \( D > 0 \) so that if \( z, w \in \overline{B} \) and \( |z - w| < t \), then \( |\phi(z) - \phi(w)| < Dt \). Let \( e = (1,0) \). Consider the case that \( \zeta = \xi = e \).

Set
\[
L = \liminf_{z \to e} \frac{1 - |\phi(z)|^2}{1 - |z|^2}.
\]

Then by the Julia-Carathéodory theory [6, pp. 174–181], \( L = \lim_{r \to 1} D_1 \phi_1(re) \).

Note that \( L \geq 1 \) by the Schwarz Lemma. For \( 0 < c < 1 \), consider the ellipsoids
\[
E_c = \left\{ z \in B : \frac{|z_1 - (1-c)|^2}{c^2} + \frac{|z_2|^2}{c} < 1 \right\}.
\]

By [6, Theorem 8.54], we have \( \phi(E_c) \subset E_{Lc} \) if \( c < 1/L \). Also note that \( E_c \subset S(e,2c) \).

Now if \( z \in Q(e,t) \), we have \( |1 - z_1| < t \), so that \( |z_1| > 1 - t \). Hence \( |z_2|^2 = |1 - z_1|^2 < 2t - t^2 \). If follows that \( (1 - 2t, z_2) \in E_{2t} \). Thus \( (1 - 2t, z_2) \in S(e,4t) \), so that \( \phi(1 - 2t, z_2) \in S(e,4tL) \).

Set \( \delta = 1/2L \). Suppose that \( 0 < t < \delta \), and \( z \in Q(e,t) \). Then \( |z - (1 - 2t, z_2)| \leq |1 - z_1| + 2t < 3t \), so that \( |\phi(z) - \phi(1 - 2t, z_2)| < D(3t) \). Thus \( |1 - \phi_1(z)| < 4tL + 3tD \), and the lemma holds with \( A = 4L + 3D \).

For the general case choose unitaries \( U \) and \( V : C^2 \to C^2 \) with \( Ue = \zeta \) and \( V^1(1,0) = e \). Apply the first part of the proof to the map \( \lambda = V \circ \phi \circ U \). There are positive numbers \( A \) and \( \delta \) so that \( \lambda(Q(e,t)) \subset S(e,At) \) for \( 0 < t < \delta \). Since \( U(Q(e,t)) = Q(\zeta, t) \) and \( V^{-1}(S(e,At)) = S(\zeta, At) \), we have \( \phi(Q(\zeta, t)) \subset S(\zeta, At) \).

Finally, note that \( A \) depends on the Lipschitz constant \( D \) and on \( L \). But \( L \leq \sup\{||\phi'(z)|| : z \in B\} \), so that both \( \delta \) and \( A \) can be chosen independent of \( \zeta \) and \( \xi \).

**Proof of Theorem 1.** Since an automorphism of \( B \) induces a bounded composition operator, we may assume that \( \phi(0) = 0 \). Fix a positive integer \( n \). Suppose that \( \xi \in \partial B \) and \( \text{card}(\phi^*|^{-1}(\xi)) \geq n \). Choose \( \zeta_1, \zeta_2, \ldots, \zeta_n \in \partial B \) so that \( \phi^*(\zeta_k) = \xi, 1 \leq k \leq n \). Choose \( A \) and \( \delta \) as in Lemma 1. Then choose \( t_0 \) with \( 0 < t_0 \leq \delta \) and so that if \( 0 < t < t_0 \), the sets \( Q(\zeta_1, t), \ldots, Q(\zeta_n, t) \) are pairwise disjoint. Thus
\[
\sigma(\phi^*|^{-1}(S(\zeta, At))) \geq \sigma \left( \bigcup_{k=1}^{n} Q(\zeta_k, t) \right) \approx nt^2.
\]

Since \( n \) is arbitrary, it is clear that \( \sigma(\phi^*|^{-1}) \) is not a Carleson measure, and the theorem is proven.

**3. Examples.**

**Example 1.** This example is a slight variant of an example shown to us by J. P. Rosay. Let
\[
\psi(z_1, z_2) = \frac{1}{2}(1 + \overline{z_1}^2 + \overline{z_2}^2, z_2(1 - z_1^2 - \overline{z_2}^2)).
\]
If \( z \in \overline{B} \), then

\[
|\psi(z)|^2 = \frac{1}{4} (1 + 2 \text{Re}(z_1^2 + z_2^2) + |z_1|^2)^2 + \frac{1}{4} (1 + 2 \text{Re}(z_1^2 + z_2^2) + |z_1|^2)^2 \\
\leq \frac{1}{4} (2 + 2|z_1^2 + z_2^2|) \leq 1.
\]

Further, \(|\psi(z)| = 1\) if and only if \( z_1^2 + z_2^2 = 1 \), in which case \( \psi(z) = e \). Thus \((\psi^*)^{-1}(e)\) is the unit circle \( C \) in the \( \text{Re } z_1, \text{Re } z_2 \) plane. Hence \( C_\psi \) is unbounded on \( H^2 \), by Theorem 1. It can be shown directly that \( \sigma(\psi^*)^{-1}(S(e,t)) \approx t^{3/2} \). We observe some additional properties of \( \psi \). Consider the complex Jacobian \( J_\psi \) on \( \overline{B} \).

It is easy to check that \( J_\psi \) vanishes only on \( C \) and on the complex line \( z_1 = 0 \). Also if \( z \) and \( w \) are in \( \overline{B} - C \) and \( \psi(z) = \psi(w) \), we have \( z_1 = \pm w_1 \) and \( z_2 = w_2 \). Thus \( \psi \) is a two-to-one map on \( \overline{B} - C \). \( \psi \) is one-to-one on \( \{z \in \overline{B}: \text{Re } z_1 > 0\} \).

EXAMPLE 2. Let \( \rho(z_1, z_2) = (1 - \sqrt{\frac{1}{2}(1 - z_1^2 - z_2^2)}, \frac{1}{2} z_2 (1 - z_1^2 - z_2^2)) \). Here \( \sqrt{\cdot} \) denotes the principal branch of the square root. Then \( \rho \) shares may properties with \( \psi \). An application of the Schwarz Lemma shows that \( |\rho_1(z)| \leq |\psi_1(z)| \) for \( z \in \overline{B} \). It follows that \( \rho(\overline{B} - C) \subset \overline{B} \). Also \( \rho(c) = \{e\} \), and \( \rho \) is two-to-one on \( \overline{B} - C \). \( \rho \) is continuous on \( \overline{B} \), but \( \rho' \) is not bounded on \( \overline{B} \).

We will show that \( C_\rho \) is compact on \( H^2 \). First we show that \( \rho(B) \) is contained in a Koranyi approach region \( D_{\alpha}(e) = \{z \in B: |1 - z_1| < (\alpha/2)(1 - |z|^2)\} \).

Let \( E_1 = \{z \in \overline{B}: |1 - z_1^2 - z_2^2| < \frac{1}{4}\} \) and \( E_2 = \overline{B} - E_1 \). Then sup\(|\rho(z)|: z \in E_2\} < 1\), so we have \( \rho(E_2) \subset D_{\alpha_0}(e) \) for some \( \alpha_0 > 0 \). If \( z \in E_1 \), then

\[
|\rho(z)|^2 \leq 1 - 2 \text{Re} \frac{1}{2}(1 - z_1^2 - z_2^2) + \frac{1}{2}|1 - z_1^2 - z_2^2| + \frac{1}{2}|1 - z_1^2 - z_2^2|^2.
\]

Thus

\[
1 - |\rho(z)|^2 \geq 2 \text{Re} \frac{1}{2}(1 - z_1^2 - z_2^2) - |1 - z_1^2 - z_2^2|.
\]

But \( \text{Re}(1 - z_1^2 - z_2^2) \geq 0 \), so

\[
\text{Re} \frac{1}{\sqrt{2}}|1 - z_1^2 - z_2^2|^{1/2} \geq \frac{1}{\sqrt{2}}|1 - z_1^2 - z_2^2|^{1/2}.
\]

Hence

\[
1 - |\rho(z)|^2 \geq |1 - z_1^2 - z_2^2|^{1/2}(1 - |1 - z_1^2 - z_2^2|^{1/2}) \geq \frac{1}{2}|1 - z_1^2 - z_2^2|^{1/2} = \frac{1}{2}|1 - \rho_1(z)|.
\]

So \( \rho(B) \subset D_\alpha(e) \), where \( \alpha = \max(\alpha_0, 4) \).

By the computation mentioned in Example 1, we have

\[
\sigma(\rho^*)^{-1}(S(e,t)) = \sigma(\psi^*)^{-1}(S(e,2t^2)) \approx t^3.
\]

By \([3, \text{Lemma } 2.1, \text{(ii)}]\), \( C_\rho \) is compact.

We now construct a biholomorphism \( \Phi \) of \( B \) into \( B \) which is a homeomorphism of \( \overline{B} \) onto \( \Phi(\overline{B}) \) and such that \( C_\Phi \) is unbounded on \( H^2 \). Let \( \psi \) be as in Example 1 and let

\[
\phi(z) = \frac{1}{25}((18 + 9z_1 - 2z_1^2 + 2z_2, 9z_2 - 4z_1z_2).
\]

We will consider the map \( \Phi = \psi \circ \phi \). Our first step is to study \( \phi \).
LEMMA 2. Let \( f(z) = 18 + 9z - 2z^2 \). If \( |z| \leq r < 1 \) and \( z \neq r \), then \( |f(z)| < f(r) \).

PROOF. Let \( z = re^{i\theta} = x + iy, 0 < r < 1 \). Then

\[
|f(z)|^2 = 18 + 9^2 r^2 + 2^2 r^4 + 2 \cdot 9 \cdot 18r^2 - 2 \cdot 2 \cdot 18(x^2 - y^2) - 2 \cdot 2 \cdot 9r^2 x
\]

\[
= 468 + 153r^2 + 4r^4 - 144(1 - x)^2 + 36x(1 - r^2)
\]

\[
\leq 468 + 153r^2 + 4r^4 - 144(1 - x^2)^2 + 36(x(1 - r^2)) = f(r)^2,
\]

with equality if and only if \( x = r \).

NOTE. \( g(z) = f(z)/25 \) is the second Taylor polynomial at \( z = 1 \) of the automorphism \( A(z) = (z + 2/3)(1 + 2z/3)^{-1} \) of the unit disc \( \Delta \). Using Lemma 2, one can see that \( g(\Delta) \subset \Delta \). Also \( g \) is univalent on \( \Delta \), \( g(1) = 1 \), and the range of \( g \) has second order contact at 1 with the unit circle.

LEMMA 3. \( \phi \) is one-to-one on \( \overline{B} \).

PROOF. Suppose that \( \phi(z) = \phi(w) \) with \( z, w \in \overline{B} \). Then \( 9z_1 - 2z_1^2 + 2z_1^2 = 9w_1 - 2w_1^2 + 2w_1^2 \), so that \( (z_1 - w_1)(9 - 2z_1 - 2w_1) = 2(w_2 - z_2)(w_2 + z_2) \). Hence \( |z_1 - w_1| \leq \frac{3}{5}|w_2 - z_2| \).

Also \( 9z_2 - 4z_1z_2 = 9w_2 - 4w_1w_2 \) so that \( (9 - 4z_1)(z_2 - w_2) = 4w_2(z_1 - w_1) \).

Thus \( |z_2 - w_2| \leq \frac{4}{5}|z_1 - w_1| \), and \( z = w \).

LEMMA 4. \( \phi(e) = e \), and \( \phi(\overline{B} - \{e\}) \subset B \).

PROOF. For \( z \in \partial B \), write \( z_1 = re^{i\theta} = x + iy \). Then \( |z_2| = \sqrt{1 - r^2} \). Lemma 2 is used in the following inequality.

\[
25|\phi(z)|^2 \leq (|18 + 9z_1 - 2z_1^2| + 2|z_2|^2)^2 + |z_2|^2|9 - 4z_1|^2
\]

\[
\leq |f(z_2)|^2 + 4f(r)(1 - r^2) + 4(1 - r^2)^2 + (1 - r^2)(81 - 72x + 16r^2)
\]

\[
= 468 + 153r^2 + 4r^4 - 144(1 - x)^2 + 36x(1 - r^2)
\]

\[
+ 4(18 + 9r - 2r^2)(1 - r^2)
\]

\[
+ 4(1 - r^2)^2 + (1 - r^2)(81 - 72x + 16r^2)
\]

\[
= 625 - 144(1 - x)^2 + 36(1 + r)(1 - r)(r - x) \leq 625
\]

since \( 36(1 + r)(1 - r)(r - x) \leq 72(1 - x)^2 \leq 144(1 - x)^2 \). Also note that equality holds if and only if \( r = x = 1 \).

The motivation behind the formula for \( \phi \) is that if \( z_1 \) and \( z_2 \) are real, then \( \phi_1(z_1, z_2) = \text{Re} g(z_1 + iz_2) \) and \( \phi_2(z_1, z_2) = \text{Im} g(z_1 + iz_2) \). Since \( g(\Delta) \subset \Delta \), one can hope that \( \phi(B) \subset B \). Further, the curve \( \phi(C) \) has second order tangency at \( e \) to \( C \).

THEOREM 2. \( \Phi(\overline{B}) \subset \overline{B}, \Phi \) is a homeomorphism of \( B \) onto \( \Phi(\overline{B}) \), and \( C_\Phi \) is unbounded on \( H^2 \).

PROOF. Since \( \phi \) and \( \psi \) both map \( \overline{B} \) into \( \overline{B} \), we have \( \Phi(\overline{B}) \subset \overline{B} \). Also \( \text{Re} \phi_1(f) > 0 \) for \( z \in \overline{B} \) and \( \psi \) is one-to-one on \( B \cap \{z: \text{Re} z_1 > 0\} \). Hence Lemmas 3 and 4 show that \( \Phi \) is a homeomorphism. It remains to show that \( C_\Phi \) is unbounded. Now

\[
\Phi_1(z) = \frac{1}{2} \left[ 1 + \frac{1}{25^2} (18 + 9z_1 - 2z_1^2 + 2z_2^2)^2 + (9z_2 - 4z_1z_2)^2 \right],
\]
and a computation shows that
\[ 1 - \Phi_1(z) = \frac{1}{1250} [144(1 - z_1)^2 + (1 - z_1^2 - z_2^2)(157 - 36z_1 + 4z_1^2 + 4z_2^2)]. \]
Thus,
\[ |1 - \Phi_1(z)| \leq \frac{1}{6} (|1 - z_1|^2 + |1 - z_1^2 - z_2^2|) \text{ for } z \in \overline{B}. \]
We will show that \( \lim_{t \to 0} (\sigma(\Phi^*)^{-1}(S(e, t))/t^2) = \infty \) so that \( \sigma(\Phi^*)^{-1} \) is not a Carleson measure.

Consider the parametrization of \( \partial B \) given by \((z_1, z_2) = (\sqrt{1 - \rho}e^{i\theta_1}, \sqrt{\rho}e^{i\theta_2}); 0 \leq \rho \leq 1, -\pi < \theta_1, \theta_2 \leq \pi. \) It is easy to check that \( d\sigma = d\rho d\theta_1 d\theta_2. \) For \( 0 < t < 1, \) let
\[ B_t = \{(\sqrt{1 - \rho}e^{i\theta_1}, \sqrt{\rho}e^{i\theta_2}): 0 < \theta_1 < t, 0 < \theta_2 < \pi, \rho < \min\{\sqrt{t}, (t/\theta_2)\}. \]
The following estimates show that \( B_t \subset (\Phi^*)^{-1}(S(e, t)). \) Suppose that \( z \in B_t. \) Then
\[ |1 - z_1|^2 = 1 + |z_1|^2 - 2|z_1| \cos \theta_1 \leq (1 - |z_1|)^2 + \theta_1^2 = (1 - \sqrt{1 - \rho})^2 + \theta_1^2 < \rho^2 + \theta_1^2 < t + t^2 < 2t. \]

Thus from (1), (2), and (3), \( |1 - \Phi_1(z)| < \frac{1}{6}(2t + 4t) = t. \)
Finally,
\[ \sigma(B_t) = \int_0^t d\theta_1 \left[ \int_0^{\sqrt{t}} d\theta_2 \int_0^{\sqrt{t}} d\rho + \int_0^{\pi} d\theta_2 \int_0^{t/\theta_2} d\rho \right] = t^2 + t \int_0^{\pi} \frac{t}{\theta_2} d\theta_2 \]
\[ = t^2 + t^2 \ln \pi + t^2 \ln \frac{1}{\sqrt{t}} \geq \frac{t^2}{2} \ln \frac{1}{t}. \]
Thus \( \sigma(\Phi^*)^{-1} \) is not a Carleson measure.

\( \Phi \) is the simplest one-to-one map we have been able to construct which induces an unbounded composition operator. However, motivated by inequalities (2) and (3), we can construct a simple (quadratic) mapping \( \Lambda \) of \( B \) into \( B \) which is two-to-one on \( \overline{B} \) and so that \( C_{\Lambda} \) is unbounded.

**Example 3.** Consider \( \Lambda(z) = \frac{1}{3} (5 + 5z_1 - z_1^2 + \frac{3}{2}z_2^2, z_2^2). \) Just as in Lemma 2, one can show that
\[ |5 + 5z_1 - z_1^2| \leq 5 + 5r - r^2 \text{ if } |z_1| = r < 1. \]
Thus if \( z \in \partial B \) and \( |z_1| = r, \) we have
\[ |\Lambda(z)|^2 \leq |\Lambda_1(z)|^2 + |\Lambda_2(z)|^2 \leq \frac{1}{9} \left( 5 + 5z_1 - z_1^2 \right) + \frac{3}{2}|z_2|^2 + \left( \frac{1}{9} (1 - r^2)^2 \right) \]
\[ \leq \frac{1}{9} \left( 5 + 5r - r^2 + \frac{3}{2}(1 - r^2) + \frac{1}{9}(1 + r)^2(1 - r)^2 \right) \]
\[ \leq \frac{1}{9} \left[ 9 - \frac{5}{2}(1 - r)^2 + \frac{4}{9}(1 - r)^2 \right] \leq 1, \]
with equality only if $z_1 = r = 1$. Thus $\Lambda(B - \{e\}) \subset B$. It is elementary to check that $\Lambda(z) = \Lambda(w)$ if and only if $z_2^2 = w_2^2$, so that $\Lambda$ is two-to-one on $B$.

$$1 - \Lambda_1(z) = 1 - \frac{1}{6} \left( 5 + 5z_1 - z_1^2 + \frac{3}{2}z_2^2 \right) = \frac{1}{18} \left[ 5(1 - z_1)^2 + 3(1 - z_1^2 - z_2^2) \right],$$

so the same argument as in the end of the proof of Theorem 2 shows that $\sigma(\Lambda^*)^{-1}$ is not a Carleson measure. We close with the following example.

**EXAMPLE 4.** Let $\phi(z) = \frac{1}{2}(1 + z_1, z_2)$ and let $\psi$ be as in Example 1. Set $\Phi = \psi \circ \phi$. Then just as for the map of Theorem 2, we have that $\Phi$ is a homeomorphism of $B$ onto $\Phi(B)$, $\Phi(e) = e$, and $\Phi(B - \{e\}) \subset B$. Also the derivative of $\Phi^{-1}$ is unbounded near $e$. We claim that $C_{\Phi}$ is bounded, even though the MacCluer-Shapiro Theorem [4, Theorem 6.4] does not apply. The proof is somewhat tedious, and we only give an outline. We must show that there is a $C > 0$ so that if $\zeta \in \partial B$ and $t > 0$, then $\sigma(\Phi^*)^{-1}(S(\zeta, t)) \leq Ct^2$. Since on the complement of a neighborhood of $e$, $|\Phi^*|$ is strictly less than 1, we need only consider $\zeta$ near 1. Then if $z \in Q(\zeta, t)$ and $t$ is small, we will also have $z$ near $e$.

If $|1 - (\Phi(z), \zeta)| < t$, then

$$|8/\zeta_1 - 4 - (1 + z_1)^2 - z_2^2| < t.$$

But

$$|4 - (1 + z_1)^2 - z_2^2| \leq |1 - z_1| \left( 3 + z_1 + (1 - |z_1|^2) \right) \leq 4|1 - z_1| + 2|1 - z_1| = 6|1 - z_1|.$$

Let $\lambda = 2\sqrt{2/\zeta_1} - 1$. Then $|\lambda| > 1$ and $\lambda$ is near 1.

$$|8/\zeta_1 - 4 - (1 + z_1)^2 - z_2^2| \geq |\lambda - z_1| \left( |\lambda + 2 + z_1| - |z_2|^2 \right) \geq 3|\lambda - z_1| - 2(1 - |z_1|) \geq |\lambda - z_1|.$$

From (4), (5), and (6) we have

$$|8/\zeta_1| |\lambda - z_1| - \sqrt{1 - |\zeta_1|^2 \sqrt{1 - |z_1|^2}} / 6|1 - z_1| < t.$$

Some computation shows that

$$\sqrt{1 - |z_1|^2 \sqrt{1 - |z_1|^2}} \leq 2 \left( 2\sqrt{2 - |\zeta_1|} - 1 - |z_1| \right) \leq 2|\lambda - z_1|.$$

Hence if $\zeta$ and $z$ are sufficiently near $e$ that $|\zeta_1| / 10 > \frac{1}{10}$ and $|1 - z_1| < \frac{1}{20}$, then from (7),

$$\frac{1}{10}|\lambda - z_1| - \frac{1}{20}|\lambda - z_1| < t.$$

Thus $z \in Q((\lambda/|\lambda|, 0), 20t)$. We can take $C = 400$.

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UNBounded Composition operators on $H^2(B_2)$


Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514