UNBOUNDED COMPOSITION OPERATORS ON $H^2(B_2)$

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ABSTRACT. Examples are given of holomorphic self-maps of the unit ball on $\mathbb{C}^2$ which induce unbounded composition operators on the Hardy space $H^2$. In particular, an example is given which is one-to-one on the closed ball. Also, a valence condition on the boundary of this ball is given which is sufficient for unboundedness of the induced composition operator.

1. Introduction. Let $B_n$ be the open unit ball in $\mathbb{C}^n$ and let $H^2 = H^2(B_n)$ be the Hardy space on $B_n$. If $\phi$ is a holomorphic mapping of $B_n$ into $B_n$, then the composition operator $C_{\phi} : f \mapsto f \circ \phi$ maps holomorphic functions on $B_n$ into holomorphic functions. If $n = 1$, it is well known that $C_{\phi}$ is a bounded operator on $H^2$ (see [5], e.g.). For $n > 1$, there are many examples (see [1, 2]) which show that $C_{\phi}$ need not be bounded. These examples exhibit a "collapsing" property on the boundary $\partial B_n$ of $B_n$. For instance, $\phi$ may map an arc on $\partial B_n$ to a point on $\partial B_n$. The main result of this note is the construction (Theorem 2) of a mapping $\Phi : B_2 \to B_2$ which is holomorphic and one-to-one on $B_2$ and such that $C_{\Phi}$ is unbounded on $H^2$. $\Phi$ is in fact a polynomial mapping.

B. MacCluer and J. Shapiro show in [4, Theorem 6.4] that if $\phi : B_n \to B_n$ is one-to-one and if the derivative of $\phi^{-1}$ is bounded on $\phi(B_n)$, then $C_{\phi}$ is bounded on $H^2$ (see also [1, Theorem 2]). Our example shows that even for one-to-one mappings, some additional hypothesis on $\phi$ must be imposed to guarantee that $C_{\phi}$ is bounded. Example 4 is also related to the above theorem. In Theorem 1 we give a valence condition on $\phi$ which is sufficient for unboundedness of $C_{\phi}$. All of our results rely on the following Carleson measure criterion for boundedness of $C_{\phi}$.

THEOREM [3]. Suppose that $\phi : B \to B$ is holomorphic and that $\mu = \sigma(\phi^*)^{-1}$. Then $C_{\phi}$ is bounded on $H^2$ if and only if there is a $C > 0$ so that $\mu(S(\zeta,t)) \leq Ct^2$ for all $\zeta \in \partial B$ and $t > 0$. In this case we say that $\mu$ is a $\sigma$-Carleson measure.

W. Rudin's book [6] will be used as a standard reference. We will restrict our attention to $B_2 = B$. For $\phi : B \to B$, write $\phi = (\phi_1, \phi_2)$. Let $\sigma$ denote surface measure on $\partial B$. If $\zeta \in \partial B$, set $\phi^*(\zeta) = \lim_{r \to 1} \phi(r\zeta)$; so $\phi^* : \partial B \to \overline{B}$. Further define $S(\zeta,t) = \{ z \in \overline{B} : |1 - \langle z, \zeta \rangle | < t \}$. Here $\langle \cdot, \cdot \rangle$ denotes the usual complex inner product in $\mathbb{C}^2$, and $t > 0$. Let $Q(\zeta,t) = S(\zeta,t) \cap \partial B$.

2. A criterion for unboundedness. In this section we prove the following.

THEOREM 1. Suppose that $\phi : B \to B$ is holomorphic on $B$ and that $\phi'$ is uniformly bounded on $B$. If $\sup \{ \text{card}(\phi^*(\xi)^{-1}(\xi)) : \xi \in \partial B \} = \infty$, then $C_{\phi}$ is unbounded on $H^2$.

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The proof of this theorem depends on the following lemma. We assume the smoothness hypothesis of Theorem 1.

**Lemma 1.** Suppose that $\phi(0) = 0$. Then there exist positive numbers $A$ and $\delta$ which satisfy the following. If $\zeta, \xi \in \partial B$ and $\phi^*(\zeta) = \xi$, then $\phi^*(Q(\zeta, t)) \subset S(\xi, At)$ for all $0 < t < \delta$.

**Proof.** $\phi$ has a continuous extension to $\overline{B}$, which we can also denote by $\phi$. In fact $\phi$ is Lipschitz on $\overline{B}$. Thus there is a $D > 0$ so that if $z, w \in \overline{B}$ and $|z - w| < t$, then $|\phi(z) - \phi(w)| < Dt$. Let $e = (1, 0)$. Consider the case that $\zeta = \xi = e$. Set

$$L = \liminf_{z \to e} \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

Then by the Julia-Carathéodory theory [6, pp. 174–181], $L = \lim_{r \to 1} D_1 \phi_1(re)$. Note that $L \geq 1$ by the Schwarz Lemma. For $0 < c < 1$, consider the ellipsoids $E_c = \left\{ z \in B : \frac{|z_1 - (1 - c)c|^2}{c^2} + \frac{|z_2|^2}{c} < 1 \right\}$.

By [6, Theorem 8.54], we have $\phi(E_c) \subset E_{Lc}$ if $c < 1/L$. Also note that $E_c \subset S(e, 2c)$.

Now if $z \in Q(e, t)$, we have $|1 - z_1| < t$, so that $|z_1| > 1 - t$. Hence $|z_2|^2 = 1 - |z_1|^2 < 2t - t^2$. If follows that $(1 - 2t, z_2) \in E_{2t}$. Thus $(1 - 2t, z_2) \in S(e, 4t)$, so that $\phi((1 - 2t, z_2)) \in S(e, 4tL)$.

Set $\delta = 1/2L$. Suppose that $0 < t < \delta$, and $z \in Q(e, t)$. Then $|z - (1 - 2t, z_2)| \leq |1 - z_1| + 2t < 3t$, so that $|\phi(z) - \phi((1 - 2t, z_2))| < D(3t)$. Thus $|1 - \phi(z)| < 4tL + 3tD$, and the lemma holds with $A = 4L + 3D$. For the general case choose unitaries $U$ and $V : C^2 \to C^2$ with $Ue = \zeta$ and $V \xi = e$. Apply the first part of the proof to the map $\lambda = V \circ \phi \circ U$. There are positive numbers $A$ and $\delta$ so that $\lambda(Q(e, t)) \subset S(e, At)$ for $0 < t < \delta$. Since $U(Q(e, t)) = Q(\zeta, t)$ and $V^{-1}(S(e, At)) = S(e, At)$, we have $\phi(Q(\zeta, t)) \subset S(\xi, At)$.

Finally, note that $A$ depends on the Lipschitz constant $D$ and on $L$. But $L \leq \sup\{|\phi'(z)| : z \in B\}$, so that both $\delta$ and $A$ can be chosen independent of $\zeta$ and $\xi$.

**Proof of Theorem 1.** Since an automorphism of $B$ induces a bounded composition operator, we may assume that $\phi(0) = 0$. Fix a positive integer $n$. Suppose that $\xi \in \partial B$ and $\text{card} (\phi^*)^{-1}(\xi) \geq n$. Choose $\zeta_1, \zeta_2, \ldots, \zeta_n \in \partial B$ so that $\phi^*(\zeta_k) = \xi$, $1 \leq k \leq n$. Choose $A$ and $\delta$ as in Lemma 1. Then choose $t_0$ with $0 < t_0 \leq \delta$ and so that if $0 < t < t_0$, the sets $Q(\zeta_1, t), \ldots, Q(\zeta_n, t)$ are pairwise disjoint. Thus

$$\sigma(\phi^*)^{-1}(S(\xi, At)) \geq \sigma \left( \bigcup_{1}^{n} Q(\zeta_k, t) \right) \approx nt^2.$$  

Since $n$ is arbitrary, it is clear that $\sigma(\phi^*)^{-1}$ is not a Carleson measure, and the theorem is proven.

3. Examples.

**Example 1.** This example is a slight variant of an example shown to us by J. P. Rosay. Let

$$\psi(z_1, z_2) = \frac{1}{2}(1 + z_1^2 + z_2^2, z_2(1 - z_1^2 - z_2^2)).$$
If $z \in \overline{B}$, then
\[ |\psi(z)|^2 = \frac{1}{4} (1 + 2 \Re(z_1^2 + z_2^2) + |z_1^2 + z_2^2|^2 + |z_1|^2(1 - 2 \Re(z_1^2 + z_2^2) + |z_1^2 + z_2^2|^2)) \leq \frac{1}{4} (2 + 2|z_1^2 + z_2^2|^2) \leq 1. \]

Further, $|\psi(z)| = 1$ if and only if $z_1^2 + z_2^2 = 1$, in which case $\psi(z) = e$. Thus $(\psi^*)^{-1}(e)$ is the unit circle $C$ in the $\Re z_1, \Re z_2$ plane. Hence $C_\psi$ is unbounded on $H^2$, by Theorem 1. It can be shown directly that $\sigma(\psi^*)^{-1}(S(e,t)) \approx t^{3/2}$. We observe some additional properties of $\psi$. Consider the complex Jacobian $J_\psi$ on $\overline{B}$. It is easy to check that $J_\psi$ vanishes only on $C$ and on the complex line $z_1 = 0$. Also if $z$ and $w$ are in $\overline{B} - C$ and $\psi(z) = \psi(w)$, we have $z_1 = \pm w_1$ and $z_2 = w_2$. Thus $\psi$ is a two-to-one map on $\overline{B} - C$. $\psi$ is one-to-one on \{z \in \overline{B}: \Re z_1 > 0\}.

EXAMPLE 2. Let $p(z_1, z_2) = (1 - \sqrt{1 - z_1^2 - z_2^2}, \frac{1}{2} z_2(1 - z_1^2 - z_2^2))$. Here $\sqrt{\cdot}$ denotes the principal branch of the square root. Then $\rho$ shares many properties with $\psi$. An application of the Schwarz Lemma shows that $|\rho_1(z)| \leq |\psi_1(z)|$ for $z \in B$. It follows that $\rho(\overline{B} - C) \subset B$. Also $\rho(c) = \{e\}$, and $\rho$ is two-to-one on $\overline{B} - C$. $\rho$ is continuous on $\overline{B}$, but $\rho'$ is not bounded on $B$.

We will show that $C_\rho$ is compact on $H^2$. First we show that $\rho(B)$ is contained in a Koranyi approach region $D_\alpha(e) = \{z \in B: |1 - z_1| < (\alpha/2)(1 - |z|^2)\}$.

Let $E_1 = \{z \in B: |1 - z_1^2 - z_2^2| < \frac{1}{4}\}$ and $E_2 = B - E_1$. Then $\sup\{|\rho(z)|: z \in E_1\} = \frac{a}{\alpha}$, so we have $\rho(E_2) \subset D_\alpha(e)$ for some $\alpha > 0$. If $z \in E_2$, then
\[ |\rho(z)|^2 \leq 1 - 2 \Re \left(1 - z_1^2 - z_2^2\right) \leq |1 - z_1^2 - z_2^2|^2. \]

Thus
\[ 1 - |\rho(z)|^2 \geq 2 \Re \left(1 - z_1^2 - z_2^2\right) - |1 - z_1^2 - z_2^2|. \]

But $\Re(1 - z_1^2 - z_2^2) \geq 0$, so
\[ \Re \left(1 - z_1^2 - z_2^2\right) \geq \frac{1}{\sqrt{2}} |1 - z_1^2 - z_2^2|^{1/2}. \]

Hence
\[ 1 - |\rho(z)|^2 \geq |1 - z_1^2 - z_2^2|^{1/2}(1 - |1 - z_1^2 - z_2^2|^{1/2}) \geq \frac{1}{2}|1 - z_1^2 - z_2^2|^{1/2} = \frac{1}{2}|1 - \rho_1(z)|. \]

So $\rho(B) \subset D_\alpha(e)$, where $\alpha = \max(\alpha_0, 4)$.

By the computation mentioned in Example 1, we have
\[ \sigma(\rho^*)^{-1}(S(e,t)) = \sigma(\psi^*)^{-1}(S(e,2t^2)) \approx t^3. \]

By [3, Lemma 2.1, (ii)], $C_\rho$ is compact.

We now construct a biholomorphism $\Phi$ of $B$ into $B$ which is a homeomorphism of $\overline{B}$ onto $\Phi(\overline{B})$ and such that $C_\Phi$ is unbounded on $H^2$. Let $\psi$ be as in Example 1 and let
\[ \phi(z) = \frac{1}{25}(18 + 9z_1 - 2z_1^2 + 2z_2^2, 9z_2 - 4z_1z_2). \]

We will consider the map $\Phi = \psi \circ \phi$. Our first step is to study $\phi$. 

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LEMMA 2. Let \( f(z) = 18 + 9z - 2z^2 \). If \(|z| \leq r < 1 \) and \( z \neq r \), then \(|f(z)| < f(r)|\).

PROOF. Let \( z = re^{i\theta} = x + iy, 0 < r < 1 \). Then
\[
|f(z)|^2 = 18 + 9r^2 + 2r^4 + 2 \cdot 9 \cdot 18x - 2 \cdot 2 \cdot 18(x^2 - y^2) - 2 \cdot 2 \cdot 9r^2 x
\]
\[
= 468 + 153r^2 + 4r^4 - 144(1 - x)^2 + 36x(1 - r^2)
\]
\[
\leq 468 + 153r^2 + 4r^4 - 144(1 - r)^2 + 36r(1 - r^2) = f(r)^2,
\]
with equality if and only if \( x = r \).

NOTE. \( g(z) = f(z)/25 \) is the second Taylor polynomial at \( z = 1 \) of the automorphism \( A(z) = (z + 2/3)(1 + 2z/3)^{-1} \) of the unit disc \( \Delta \). Using Lemma 2, one can see that \( g(\Delta) \subset \Delta \). Also \( g \) is univalent on \( \Delta \), \( g(1) = 1 \), and the range of \( g \) has second order contact at 1 with the unit circle.

LEMMA 3. \( \phi \) is one-to-one on \( B \).

PROOF. Suppose that \( \phi(z) = \phi(w) \) with \( z, w \in \overline{B} \). Then \( 9z_1 - 2z_1^2 + 2z_1^2 = 9w_1 - 2w_1^2 + 2w_1^2 \), so that \( (z_1 - w_1)(9 - 2z_1 - 2w_1) = 2(w_2 - z_2)(w_2 + z_2) \). Hence \( |z_1 - w_1| \leq \frac{1}{3}|w_2 - z_2| \).

Also \( 9z_2 - 4z_1 z_2 = 9w_2 - 4w_1 w_2 \) so that \( (9 - 4z_1)(z_2 - w_2) = 4w_2(z_1 - w_1) \). Thus \( |z_2 - w_2| \leq \frac{4}{3}|z_1 - w_1| \), and \( z = w \).

LEMMA 4. \( \phi(e) = e \), and \( \phi(\overline{B} - \{e\}) \subset B \).

PROOF. For \( z \in \partial B \), write \( z_1 = re^{i\theta} = x + iy \). Then \( |z_2| = \sqrt{1 - r^2} \). Lemma 2 is used in the following inequality.
\[
25|\phi(z)|^2 \leq (|18 + 9z_1 - 2z_1^2| + 2|z_2|^2)^2 + |z_2|^2|9 - 4z_1|^2
\]
\[
\leq |f(z_1)|^2 + 4f(r)(1 - r^2) + 4(1 - r^2)^2 + (1 - r^2)(81 - 72x + 16r^2)
\]
\[
= 468 + 153r^2 + 4r^4 - 144(1 - x)^2 + 36x(1 - r^2)
\]
\[
+ 4(18 + 9r - 2r^2)(1 - r^2)
\]
\[
+ 4(1 - r^2)^2 + (1 - r^2)(81 - 72x + 16r^2)
\]
\[
= 625 - 144(1 - x)^2 + 36(1 + r)(1 - r)(r - x) \leq 625
\]
since \( 36(1 + r)(1 - r)(r - x) \leq 72(1 - x)^2 \leq 144(1 - x)^2 \). Also note that equality holds if and only if \( r = x = 1 \).

The motivation behind the formula for \( \phi \) is that if \( z_1 \) and \( z_2 \) are real, then \( \phi_1(z_1, z_2) = \Re g(z_1 + iz_2) \) and \( \phi_2(z_1, z_2) = \Im g(z_1 + iz_2) \). Since \( g(\Delta) \subset \Delta \), one can hope that \( \phi(B) \subset B \). Further, the curve \( \phi(C) \) has second order tangency at \( e \) to \( C \).

THEOREM 2. \( \Phi(\overline{B}) \subset \overline{B}, \Phi \) is a homeomorphism of \( B \) onto \( \Phi(\overline{B}) \), and \( C_\Phi \) is unbounded on \( H^2 \).

PROOF. Since \( \phi \) and \( \psi \) both map \( \overline{B} \) into \( \overline{B} \), we have \( \Phi(\overline{B}) \subset \overline{B} \). Also \( \Re \phi_1(f) > 0 \) for \( z \in \overline{B} \) and \( \psi \) is one-to-one on \( B \cap \{ z : \Re z_1 > 0 \} \). Hence Lemmas 3 and 4 show that \( \Phi \) is a homeomorphism. It remains to show that \( C_\Phi \) is unbounded. Now
\[
\Phi_1(z) = \frac{1}{2} \left[ 1 + \frac{1}{25}((18 + 9z_1 - 2z_1^2 + 2z_2^2)^2 + (9z_2 - 4z_1 z_2)^2) \right],
\]
and a computation shows that

\[ 1 - \Phi_1(z) = \frac{1}{1250} \left[ 144(1 - z_1)^2 + (1 - z_1^2 - z_2^2)(157 - 36z_1 + 4z_1^2 + 4z_2^2) \right]. \]

Thus,

(1) \[ |1 - \Phi_1(z)| \leq \frac{1}{6} (|1 - z_1|^2 + |1 - z_1^2 - z_2^2|) \quad \text{for } z \in \overline{B}. \]

We will show that \( \lim_{t \to 0} (\sigma(\Phi^*)^{-1}(S(e,t))/t^2) = \infty \) so that \( \sigma(\Phi^*)^{-1} \) is not a Carleson measure.

Consider the parametrization of \( \partial B \) given by \((z_1, z_2) = (\sqrt{1 - \rho e^{i\theta_1}}, \sqrt{\rho e^{i\theta_2}}); 0 \leq \rho \leq 1, -\pi < \theta_1, \theta_2 \leq \pi. \) It is easy to check that \( d\sigma = d\rho d\theta_1 d\theta_2. \) For \( 0 < t < 1, \)

\[ B_t = \{(\sqrt{1 - \rho e^{i\theta_1}}, \sqrt{\rho e^{i\theta_2}}); 0 < \theta_1 < t, 0 < \theta_2 < \pi, \rho < \min\{\sqrt{t}, (t/\theta_2)\}. \]

The following estimates show that \( B_t \subset (\Phi^*)^{-1}(S(e,t)). \) Suppose that \( z \in B_t. \) Then

(2) \[ |1 - z_1|^2 = 1 + |z_1|^2 - 2|z_1| \cos \theta_1 \leq (1 - |z_1|)^2 + \theta_1^2 \]

\[ = (1 - \sqrt{1 - \rho})^2 + \theta_1^2 \leq \rho^2 + \theta_1^2 < t + t^2 < 2t. \]

Also

(3) \[ |1 - z_1^2 - z_2^2| = |1 - (1 - \rho)e^{2i\theta_1} - \rho e^{2i\theta_2}| \leq \rho |1 - e^{2i\theta_2}| + (1 - \rho)|1 - e^{2i\theta_1}| \leq 2\rho \theta_2 + 2\theta_1 < 2t + 2t = 4t. \]

Thus from (1), (2), and (3), \( |1 - \Phi_1(z)| \leq \frac{1}{6}(2t + 4t) = t. \)

Finally,

\[ \sigma(B_t) = \int_0^t d\theta_1 \left[ \int_0^{\sqrt{t}} d\theta_2 \int_0^{\sqrt{t}} d\rho + \int_0^{\pi} d\theta_2 \int_0^{t/\theta_2} d\rho \right] = t^2 + t \int_0^{\pi} \frac{t}{\theta_2} d\theta_2 \]

\[ = t^2 + t^2 \ln \pi + t^2 \ln \frac{1}{\sqrt{t}} \geq \frac{t^2}{2} \ln \frac{1}{t}. \]

Thus \( \sigma(\Phi^*)^{-1} \) is not a Carleson measure.

\( \Phi \) is the simplest one-to-one map we have been able to construct which induces an unbounded composition operator. However, motivated by inequalities (2) and (3), we can construct a simple (quadratic) mapping \( \Lambda \) of \( B \) into \( B \) which is two-to-one on \( \overline{B} \) and so that \( C_{\Lambda} \) is unbounded.

**Example 3.** Consider \( \Lambda(z) = \frac{1}{9} (5 + 5z_1 - z_1^2 + \frac{3}{2} z_2^2, z_2^2). \) Just as in Lemma 2, one can show that

\[ |5 + 5z_1 - z_1^2| \leq 5 + 5r - r^2 \quad \text{if } |z_1| = r < 1. \]

Thus if \( z \in \partial B \) and \( |z_1| = r, \) we have

\[ |\Lambda(z)|^2 \leq |\Lambda_1(z)| + |\Lambda_2(z)|^2 \leq \frac{1}{9} (|5 + 5z_1 - z_1^2| + \frac{3}{2}|z_2|^2) + \frac{1}{91}(1 - r^2)^2 \]

\[ \leq \frac{1}{9} [5 + 5r - r^2 + \frac{3}{2}(1 - r^2) + \frac{1}{9}(1 + r)^2(1 - r^2)] \]

\[ \leq \frac{1}{9} [9 - \frac{5}{2}(1 - r^2) + \frac{4}{9}(1 - r^2)] \leq 1, \]
with equality only if $z_1 = r = 1$. Thus $\Lambda(\overline{B} - \{e\}) \subset B$. It is elementary to check that $\Lambda(z) = \Lambda(w)$ if and only if $z_2^2 = w_2^2$, so that $\Lambda$ is two-to-one on $\overline{B}$.

$$1 - \Lambda_1(z) = 1 - \frac{1}{9} (5 + 5z_1 - z_1^2 + \frac{3}{2} z_2^2)$$

$$= \frac{1}{18} [5(1 - z_1)^2 + 3(1 - z_1^2 - z_2^2)],$$

so the same argument as in the end of the proof of Theorem 2 shows that $\sigma(\Lambda^*)^{-1}$ is not a Carleson measure. We close with the following example.

**Example 4.** Let $\phi(z) = \frac{1}{2} (1 + z_1, z_2)$ and let $\psi$ be as in Example 1. Set $\Phi = \psi \circ \phi$. Then just as for the map of Theorem 2, we have that $\Phi$ is a homeomorphism of $\overline{B}$ onto $\overline{\Phi(B)}$, $\Phi(e) = e$, and $\Phi(\overline{B} - \{e\}) \subset B$. Also the derivative of $\Phi^{-1}$ is unbounded near $e$. We claim that $C_\Phi$ is bounded, even though the MacCluer-Shapiro Theorem [4, Theorem 6.4] does not apply. The proof is somewhat tedious, and we only give an outline. We must show that there is a $C > 0$ so that if $z \in \partial B$ and $t > 0$, then $\sigma(\Phi^*)^{-1}(S(z, t)) \leq Ct^2$. Since on the complement of a neighborhood of $e$, $|\Phi^*|$ is strictly less than 1, we need only consider $z$ near 1. Then if $z \in Q(\zeta, t)$ and $t$ is small, we will also have $z$ near $e$.

If $|1 - (\Phi(z), \zeta)| < t$, then

$$|\frac{8}{|s_1|} s_1 - 4 - (1 + z_1)^2 - z_2^2| < t.$$ 

But

$$|4 - (1 + z_1)^2 - z_2^2| \leq |1 - z_1||3 + z_1| + (1 - |z_1|^2)$$

$$\leq 4|1 - z_1| + 2|1 - z_1| = 6|1 - z_1|.$$ 

Let $\lambda = 2\sqrt{2/|s_1| - 1}. Then $|\lambda| > 1$ and $\lambda$ is near 1.

$$|8/\zeta_1 - 4 - (1 + z_1)^2 - z_2^2| \geq |\lambda - z_1| |\lambda + 2 + z_1| - |z_2|^2$$

$$\geq 3|\lambda - z_1| - 2(1 - |z_1|) \geq |\lambda - z_1|.$$ 

From (4), (5), and (6) we have

$$|\frac{s_1}{8}|^2 |\lambda - z_1| - \frac{\sqrt{1 - |s_1|^2} \sqrt{1 - |z_1|^2}}{16} |1 - z_1| < t.$$ 

Some computation shows that

$$\sqrt{1 - |s_1|^2} \sqrt{1 - |z_1|^2} \leq 2 \left(2\sqrt{2/|s_1|} - 1 - |z_1|\right) \leq 2|\lambda - z_1|.$$ 

Hence if $\zeta$ and $z$ are sufficiently near $e$ that $|s_1|/8 > \frac{1}{10}$ and $|1 - z_1| < \frac{1}{20}$, then from (7),

$$\frac{1}{10}|\lambda - z_1| - \frac{1}{20}|\lambda - z_1| < t.$$ 

Thus $z \in Q((\lambda/|\lambda|, 0), 20t)$. We can take $C = 400$.

**References**


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