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# ON THE ERGODIC HILBERT TRANSFORM FOR LAMPERTI OPERATORS

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ABSTRACT. This paper is devoted to the proof of almost everywhere existence of the ergodic Hilbert transform for a class of Lamperti operators.

**1. Introduction.** Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $T$ a bounded linear operator on $L_p = L_p(X, \mathcal{F}, \mu)$, $1 \leq p < \infty$. $T$ is called a *Lamperti operator* if it maps functions with disjoint support to the same. It is known (see e.g. [**7**, **9**]) that Lamperti operators include $L_p$ isometries, $p \neq 2$, and positive $L_2$ isometries. If $T$ is invertible, then by considering known results about the classical discrete Hilbert transform, the following question arises: Does the limit

$$Hf = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \qquad (f \in L_p) \tag{1}$$

exist in any sense? (Here the prime means that the term with zero denominator is omitted.) Under the assumption that $T$ is induced by an invertible measure preserving transformation, Cotlar [**3**] proved that if $1 < p < \infty$, then (1) exists almost everywhere and in the strong operator topology; if $p = 1$, then (1) exists almost everywhere (see also Calderón [**2**] and Petersen [**10**]). In this paper we shall assume that $T$ is an invertible Lamperti operator such that $\sup\{\|T^n\|_p \colon -\infty < n < \infty\} < \infty$, and prove that if $1 < p < \infty$, then (1) exists almost everywhere and in the strong operator toplology; if $p = 1$, then, under the additional hypothesis that $\sup\{\|T^n\|_\infty \colon -\infty < n < \infty\} < \infty$, (1) exists almost everywhere. It is interesting to note that the author proved in [**11**] that if $T$ is an invertible *positive* operator on $L_p$, $1 < p < \infty$, such that $\sup\{\|T^n\|_p \colon -\infty < n < \infty\} < \infty$ then (1) exists almost everywhere and in the strong operator topology.

**2. Results.** The following maximal theorem is a key lemma to prove the almost everywhere existence of the ergodic Hilbert transform.

THEOREM 1. *Let $T$ be an invertible Lamperti operator on $L_p$, $1 < p < \infty$, such that* $\sup\{\|T^n\|_p \colon -\infty < n < \infty\} = M < \infty$. *Define the ergodic maximal Hilbert transform $H^*$, associated with $T$, as*

$$H^* f = \sup_{n\geq 1} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \right|. \tag{2}$$

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*Then there exists a constant $C > 0$, depending only on $M$, such that $\|H^* f\|_p \leq C\|f\|_p$ for all $f \in L_p$.*

PROOF. It follows from Lamperti [9] (cf. also Kan [8]) that there exists a $\sigma$-endomorphism $\Phi$ of the Boolean $\sigma$-algebra $\mathcal{F}(\mu)$, associated with the measure space $(X, \mathcal{F}, \mu)$, and a measurable function $h$ on $X$ such that

$$Tf = h \cdot \Phi f \quad \text{for all } f \in L_p,$$

where we denote by the same letter $\Phi$ the linear operator on the space of measurable functions induced by the $\sigma$-endomorphism. Since $T$ is invertible by hypothesis, $|h| > 0$ on $X$ and $\Phi$ is one-to-one and onto from $\mathcal{F}(\mu)$ to $\mathcal{F}(\mu)$. Thus we can define

$$Uf = \frac{1}{\Phi^{-1}h}\Phi^{-1}f \qquad (f \in L_p).$$

Then

$$UTf = \frac{1}{\Phi^{-1}h}\Phi^{-1}(Tf) = \frac{1}{\Phi^{-1}h}(\Phi^{-1}h)f = f,$$

and so $U = T^{-1}$. Put $h_1 = h$, $h_0 = 1$, $h_{-1} = 1/\Phi^{-1}h$, $h_n = h_1 \cdot \Phi h_{n-1}$, and $h_{-n} = h_{-1} \cdot \Phi^{-1}h_{-n+1}$ $(n \geq 2)$. It follows that

$$T^j f = h_j \cdot \Phi^j f \qquad (j = 0, \pm 1, \dots) \tag{3}$$

and

$$h_{j+k} = h_j \cdot \Phi^j h_k \qquad (j, k = 0, \pm 1, \dots); \tag{4}$$

in fact (4) follows from the equalities

$$h_{j+k} \cdot \Phi^{j+k} f = T^{j+k} f = T^j(h_k \Phi^k f) = h_j \cdot \Phi^j h_k \cdot \Phi^{j+k} f.$$

For an integer $K \geq 1$, define the truncated maximal operator $H_K^*$ as

$$H_K^* f = \max_{1 \leq n \leq K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \right| \quad \left( = \max_{1 \leq n \leq K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} h_k \cdot \Phi^k f \right| \right).$$

Since $\|T^j\|_p \leq M$ for all $j$, it then follows that

$$\begin{aligned}
\|H_K^* f\|_p^p &\leq \frac{M^p}{2L+1} \int \sum_{j=-L}^{L} |T^j H_K^* f|^p \, d\mu \\
&= \frac{M^p}{2L+1} \int \sum_{j=-L}^{L} |h_j \cdot \Phi^j (H_K^* f)|^p \, d\mu \\
&= \frac{M^p}{2L+1} \int \sum_{j=-L}^{L} \left| h_j \left[ \max_{1 \leq n \leq K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} \Phi^j h_k \cdot \Phi^{j+k} f \right| \right] \right|^p d\mu \\
&= \frac{M^p}{2L+1} \int \sum_{j=-L}^{L} \left[ \max_{1 \leq n \leq K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} h_{j+k} \cdot \Phi^{j+k} f \right| \right]^p d\mu \\
&\leq \frac{M^p}{2L+1} C^p \int \sum_{j=-L-K}^{L+K} |h_j \cdot \Phi^j f|^p \, d\mu \\
&\leq \frac{M^p}{2L+1} C^p 2(L+K+1) M^p \|f\|_p^p,
\end{aligned}$$

where the second inequality from the bottom is due to a known result about the classical discrete Hilbert transform (see e.g. Hunt, Muckenhoupt, and Wheeden [**6**]); letting $L \to \infty$, we get

$$\|H_K^* f\|_p^p \leq M^{2p} C^p \|f\|_p^p.$$

This completes the proof, since $H_K^* f \uparrow H^* f$.

THEOREM 2. *Let $T$ be as in Theorem 1. Then the limit (1) exists almost everywhere and in the strong operator toplology.*

PROOF. See the proof of Theorem 2 in [**11**].

THEOREM 3. *Let $T$ be an invertible Lamperti operator on $L_1$ such that*

$$\sup\{\|T^n\|_1 \colon -\infty < n < \infty\} = M_1 < \infty \tag{5}$$

*and*

$$\sup\{\|T^n\|_\infty \colon -\infty < n < \infty\} = M_\infty < \infty. \tag{6}$$

*Then the limit (1) exists almost everywhere.*

To prove this, we need the following lemma.

LEMMA. *Let $T$ be as in Theorem 3. Then there exists a constant $C > 0$ such that for any $f \in L_1$ and $\varepsilon > 0$*

$$\mu\{H^* f > \varepsilon\} \leq \frac{C}{\varepsilon}\|f\|_1. \tag{7}$$

PROOF. Since $T^j f = h_j \cdot \Phi^j f$ by (3), it may be assumed without loss of generality that $\mathcal{F}$ is generated by a countable collection of sets in $X$. Then, by considering a finite equivalent measure and making use of the isomorphism of a separable nonatomic normalized measure algebra with the measure algebra of the unit interval (cf. [**5**, p. 173]), we observe that $\mathcal{F}(\mu)$ is $\sigma$-isomorphic to the Boolean $\sigma$-algebra $\mathcal{B}(\lambda)$ associated with a certain measure space $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel subset of $[0,1]$ and $\lambda$ is a finite measure. Thus we may assume that $X = [0,1]$ and $\mathcal{F}(\mu) = \mathcal{B}(\lambda)$. Since $\Phi$ is one-to-one and onto from $\mathcal{F}(\mu)$ to $\mathcal{F}(\mu)$, it follows (see e.g. [**1**, pp. 69–73]) that there exists a one-to-one and onto mapping $S$ from $X$ to $X$ such that

(i) $A \in \mathcal{B}$ if and only if $SA \in \mathcal{B}$,
(ii) $\lambda A > 0$ if and only if $\lambda(SA) > 0$,
(iii) for each integer $j$ and measurable function $f$, $\Phi^j f = f \circ S^j$.

It follows from (4) that for almost all $x \in X$

$$h_{j+k}(x) = h_j(x) h_k(S^j x) \qquad (j, k = 0, \pm 1, \dots). \tag{8}$$

Therefore for almost all $x \in X$

$$\begin{aligned} |(T^j H_K^*)(x)| &= |h_j(x) H_K^* f(S^j x)| \\ &= \max_{1 \leq n \leq K} \left| {\sum_{k=-n}^{n}}' \frac{1}{k} h_j(x) h_k(S^j x) f(S^{j+k} x) \right| \\ &= \max_{1 \leq n \leq K} \left| {\sum_{k=-n}^{n}}' \frac{1}{k} h_{j+k}(x) f(S^{j+k} x) \right|. \end{aligned}$$

From this and the fact that $\|T^{-j}\|_\infty = \|1/h_j\|_\infty \le M_\infty$ it follows that

$$H_K^* f(S^j x) \le M_\infty \max_{1\le n\le K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} h_{j+k}(x) f(S^{j+k}x) \right| .$$

Since $\|T^j\|_1 \le M_1$ and $\|T^j\|_\infty = \|h_j\|_\infty \le M_\infty$ for all $j$, we then have

$$\begin{aligned}
(2L+1)\mu\{H_K^* f > \varepsilon\} &\le M_1 \sum_{j=-L}^{L} \|T^j \chi_{\{H_K^* f>\varepsilon\}}\|_1 \\
&= M_1 \int \sum_{j=-L}^{L} |h_j(x)| \chi_{\{H_K^* f>\varepsilon\}}(S^j x)\, d\mu(x) \\
&= M_1 \int \sum_{\{j:\ -L\le j\le L, H_K^* f(S^j x)>\varepsilon\}} |h_j(x)|\, d\mu(x) \\
&\le M_1 \int \sum_{\{j:\ -L\le j\le L, \varepsilon/M_\infty < \max_{1\le n\le K} |\sum_{k=-n}^{\prime n} (1/k) h_{j+k}(x) f(S^{j+k}x)|\}} |h_j(x)|\, d\mu(x) \\
&\le \frac{M_1 M_\infty^2}{\varepsilon} C \int \sum_{j=-L-K}^{L+K} |h_j(x) f(S^j x)|\, d\mu(x) \\
&\le \frac{M_1 M_\infty^2}{\varepsilon} C 2(L+K+1) M_1 \|f\|_1,
\end{aligned}$$

where the second inequality from the bottom is due to a known result about the classical discrete Hilbert transform (see e.g. [**6**]). Letting $L \to \infty$, we get

$$\mu\{H_K^* f > \varepsilon\} \le \frac{M_1^2 M_\infty^2}{\varepsilon} C \|f\|_1,$$

and the proof is completed.

PROOF OF THEOREM 3. The Riesz convexity theorem implies $\|T^n\|_p \le \max\{M_1, M_\infty\}$ for all integers $n$ and $p$, $1 < p < \infty$. Thus, by Theorem 2, the limit (1) exists almost eveywhere for all $f \in L_1 \cap L_p$. Since $H^* f < \infty$ almost everywhere for all $f \in L_1$ by the above lemma, and since $L_1 \cap L_p$ is a dense subspace of $L_1$, Banach's convergence theorem (see e.g. [**4**, p. 332]) establishes Theorem 3.

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