BOUNDARY VALUE PROBLEMS
FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Conditions sufficient to guarantee existence and uniqueness of solutions to multipoint boundary value problems for the first-order differential equation \( y' = h(t, y) \) are given when \( h \) fails to be Lipschitz along a solution of \( y' = h(t, y) \) and the initial-value problem thus has nonunique solutions.

It is well known that the initial value problem for the first-order differential equation \( y' = h(t, y) \) does not generally have a unique solution if \( h \) fails to be Lipschitz in \( y \). This raises the possibility, for non-Lipschitz \( h \), of well-posedness of problems that would be overspecified if \( h \) satisfied a Lipschitz condition; in particular, of the reasonableness of problems that would normally be associated with higher-order equations [1, 3]. Here we examine existence and uniqueness of solutions to two- and multi-point boundary value problems when \( y' = h(t, y) \) has a solution \( y = a(t) \) along which \( h \) fails to be Lipschitz. Making the change of variable \( y - a(t) \rightarrow y \), we may without loss of generality assume that \( h \) vanishes when \( y = 0 \) and that \( h \) is not Lipschitz in any neighborhood of \( y = 0 \). We treat first the case of separable variables \( h(t, y) = g(t)f(y) \) and then use a comparison theorem to treat the more general case.

Let \( b > 0 \) and consider the boundary value problem

\[
\begin{align*}
y' &= g(t)f(y), & y(0) = A, & y(b) = B \\
\end{align*}
\]

when \( A > 0, B > 0, g > 0, \) and \( yf(y) < 0 \) for \( y \neq 0 \). Then \( y' < 0 \) for \( y > 0 \) and \( y' > 0 \) for \( y < 0 \), from which it is clear that the problem has no solution. Similarly there is no solution if \( yf(y) > 0 \) for \( y \neq 0 \). Thus if solutions are to exist in general, then the right-hand side cannot change sign.

It is interesting to note that, since the two-point boundary value problem

\[
y'' = \lambda y^{1/2}, \quad y(0) = y(b) = A > 0
\]

has a unique nonnegative solution for each \( b > 0 \) and this solution vanishes on an interval when \( \lambda \) is sufficiently large [2], the following considerations do not extend to higher-order equations.

THEOREM 1. Let \( A, B \geq 0 \) and let \( f \geq 0 \) be continuous on \([-A, B]\) and vanish on \((-A, B)\) precisely at \( \alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} = 0 < \cdots < \alpha_n \). Let \( g > 0 \) be continuous on \([0, b]\). Then there is no solution of the two-point boundary value problem (1) unless the improper integral

\[
\int_{-A}^{B} \frac{d\zeta}{f(\zeta)}
\]
exists. If this integral converges, (1) has continuously differentiable solutions if and only if there exist \(T_1\) and \(T_2\) in \([0, b]\) satisfying

\[
\int_0^{T_1} g(s) ds = \int_A^0 \frac{d\zeta}{f(\zeta)}, \quad \int_{T_2}^b g(s) ds = \int_0^B \frac{d\zeta}{f(\zeta)},
\]

and

\[
T_1 \leq T_2.
\]

PROOF. We show first the necessity of these conditions. Suppose \(y(t)\) is a solution of (1) and set \(\alpha_0 = -A, \alpha_{n+1} = B\) (\(f\) may or may not vanish at these points). On each \((\alpha_i, \alpha_{i+1})\), \(f \neq 0\). Define

\[
t_i = \sup\{t: y(t) < \alpha_i\}, \quad i = 1, 2, \ldots, n + 1, \\
s_i = \inf\{t: y(t) > \alpha_i\}, \quad i = 0, 1, \ldots, n.
\]

Then \(f(y(t)) \neq 0\) on \((s_i, t_{i+1})\). Let \(\varepsilon, \delta > 0\) be sufficiently small. Dividing the differential equation by \(f(y(t))\), integrating from \(s_i + \varepsilon\) to \(t_{i+1} - \delta\), and passing to the limit as \(\varepsilon, \delta \to 0\) yields

\[
\int_{s_i}^{t_{i+1}} \frac{d\zeta}{f(\zeta)} = \int_{y(s_i)}^{y(t_{i+1})} \frac{y'(t) dt}{f(y(t))} = \int_{s_i}^{t_{i+1}} g(t) dt, \quad i = 0, \ldots, n.
\]

Thus the improper integral (2) converges. It also follows that

\[
\int_0^A \frac{d\zeta}{f(\zeta)} = \sum_{i=0}^k \int_{s_i}^{t_{i+1}} g(t) dt \leq \int_0^{t_{k+1}} g(t) dt;
\]

hence there exists a \(T_1 \leq t_{k+1}\) such that the first equality of (3) holds. In the same way, there exists \(T_2 \geq t_{k+1}\) such that the second equality of (3) obtains. The necessity of the hypotheses follows.

To show sufficiency, let \(\hat{y}\) be the maximal solution of the initial value problem \(y' = g(t)f(y), y(0) = -A\); we shall show that \(\hat{y}(T_1) = 0\). Let \(\alpha_i, i = 0, \ldots, n + 1\), be as before; then convergence of the improper integral (2) guarantees the existence of the improper integrals

\[
\int_{\alpha_i}^{\alpha_{i+1}} \frac{d\zeta}{f(\zeta)}, \quad i = 0, \ldots, n.
\]

Since \(T_1\) satisfying (3) exists, there also exist \(t_1 < t_2 < \cdots < t_{k+1} \leq T_1\) such that

\[
\sum_{i=1}^j \int_{\alpha_{i-1}}^{\alpha_i} \frac{d\zeta}{f(\zeta)} = \int_0^{t_j} g(s) ds \quad (j = 1, \ldots, k + 1);
\]

set \(t_0 = 0\).

Let \(y_i\) \((i = 0, \ldots, k)\) be the maximal solution of the initial value problem

\[
y_i'(t) = g(t)f(y_i(t)), \quad y_i(t_i) = \alpha_i;
\]

we shall show first that \(y_i\) is defined on \([t_i, t_{i+1}]\) and \(y_i(t_{i+1}) = \alpha_{i+1}\). To this end let \(y_{i, \varepsilon}\), for all sufficiently small \(\varepsilon > 0\), be the maximal solution of

\[
y_i', \varepsilon = g(t)f(y_{i, \varepsilon}) + \varepsilon, \quad y_{i, \varepsilon}(t_i) = \alpha_i + \varepsilon.
\]
If \( y_i \) is defined on \([t_i, s_i]\), then for any \( \delta > 0 \), \( y_{i, \varepsilon} \) exists for \( t_i \leq t \leq s_i - \delta \) for all sufficiently small \( \varepsilon \), and \( y_{i, \varepsilon} \downarrow y_i \) as \( \varepsilon \downarrow 0 \) [4]. Suppose that \( y_i(t) \equiv \alpha_i \) on \([t_i, \bar{t}_i]\) with \( \bar{t}_i > t_i \) and \( t_i \) small enough that \( \theta \) defined by

\[
\int_{\alpha_i}^{t_i} \frac{d\varsigma}{f(\varsigma)} = \int_{\bar{t}_i}^{t_i} g(t) \, dt
\]

exists and satisfies \( \theta < \alpha_{i+1} \); this is guaranteed by the existence of (2). Since \( g > 0 \), \( \theta > \alpha_i \). For \( \varepsilon_0 > 0 \) sufficiently small, the maximal solution \( y_{i, \varepsilon} \) of (5) exists on \([t_i, \bar{t}_i]\) for \( 0 < \varepsilon \leq \varepsilon_0 \). By further reducing \( \bar{t}_i \) and \( \theta \), if necessary, we may assume that \( y_{i, \varepsilon}(t) < \alpha_{i+1} \) for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( t \in [t_i, \bar{t}_i] \). Then \( \alpha_i < y_{i, \varepsilon}(t) < \alpha_{i+1} \) on \((t_i, \bar{t}_i)\), so \( f(y_{i, \varepsilon}(t)) > 0 \) there, and we get from (5) that

\[
\int_{t_i}^{\bar{t}_i} \frac{y_{i, \varepsilon}'(t)}{f(y_{i, \varepsilon}(t))} \, dt > \int_{t_i}^{\bar{t}_i} g(t) \, dt
\]

and hence that

\[
\int_{\alpha_i}^{y_{i, \varepsilon}(t_i)} \frac{d\varsigma}{f(\varsigma)} > 0
\]

Thus \( y_{i, \varepsilon}(t_i) \geq \theta > \alpha_i \), a contradiction. We have therefore shown that \( y_i(t) = \alpha_i \) only for \( t = t_i \).

Suppose now that \( \alpha_i < y_i(t) < \alpha_{i+1} \) for \( t_i < t < s \). Then \( f(y_i(t)) > 0 \), so we get that

\[
\int_{y_i(t_i+\varepsilon)}^{y_i(s-\delta)} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i+\varepsilon}^{s-\delta} g(t) \, dt
\]

for sufficiently small \( \varepsilon, \delta > 0 \). Passing to the limit as \( \varepsilon \to 0, \delta \to 0 \), we have that

\[
\int_{\alpha_i}^{y_i(s-\delta)} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{s} g(t) \, dt.
\]

But we have

\[
\int_{\alpha_i}^{\alpha_{i+1}} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{t_{i+1}} g(t) \, dt.
\]

Therefore if \( s < t_{i+1} \), then \( y_i(s-) < \alpha_{i+1} \) and \( y_i \) can be extended to the right of \( s \). So \( s \geq t_{i+1} \). But the unique solution to

\[
\int_{\alpha_i}^{\theta} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{t_{i+1}} g(s) \, ds
\]

is \( \theta = \alpha_{i+1} \). Therefore \( y_i(t_{i+1}) = \alpha_{i+1} \).

We have shown that on each interval \([\alpha_i, \alpha_{i+1}]\) the maximal solution \( y_i \) of \( y'_i = g(t)f(y_i), \ y_i(t_i) = \alpha_i \) exists and satisfies \( y_i(t_{i+1}) = \alpha_{i+1} \). Since \( f(\alpha_i) = 0 \) for \( i = 1, \ldots, k \), we have easily that \( y'_i(\alpha_i-0) = y'_i(\alpha_{i+1}+0) = 0 \) (\( i = 1, \ldots, k \)) and \( y'_0(\alpha_1-0) = 0 \). It follows that \( \dot{y}_1 \) defined by

\[
\dot{y}_1(t) = y_i(t), \quad t_i \leq t < t_{i+1},
\]

is the maximal solution of \( y' = g(t)f(y), \ y(0) = -A \) on \([0, T_1)\) and \( \dot{y}_1(T_1-) = \dot{y}'_1(T_1-) = 0 \).
In the same way we conclude there exists the minimal solution of
\[ \dot{y}_2(t) = g(t)\cdot f(\dot{y}_2(t)), \quad \dot{y}_2(b) = B \]
defined on \([T_2, b]\) and satisfying \(\dot{y}_2(T_2^+) = \dot{y}_2'(T_2^+) = 0\). Therefore, since \(T_1 < T_2\),
\[
y(t) = \begin{cases} 
\dot{y}_1(t), & 0 \leq t < T_1, \\
0, & T_1 \leq t \leq T_2, \\
\dot{y}_2(t), & T_2 < t \leq b,
\end{cases}
\]
defines a classical solution of (1), proving the theorem.

**Corollary 1.** If \(f\) and \(g\) are as in the theorem and \(\int_0^\infty g = \int_{-\infty}^0 g = \infty\), then the two-point boundary problem (1) has a solution for each \(A, B \geq 0\) provided \(b\) is sufficiently large (depending on \(A\) and \(B\)).

The following result is an easy extension to the multi-point boundary value problem. It is interesting to compare it with corresponding results for higher-order Lipschitz equations [3].

**Corollary 2.** Let \(f\) and \(g\) satisfy the hypotheses of the theorem. Let \(C_i \in (\alpha_i, \alpha_{i+1})\) for \(i = 1, \ldots, n-1\) and \(0 < t_1 < \cdots < t_{n-1} < b\) be given. Then the boundary value problem
\[
y' = g(t)\cdot f(y), \quad y(0) = -A, \quad y(t_1) = C_1, \ldots, y(t_{n-1}) = C_{n-1}, \quad y(b) = B
\]
has a solution if and only if there exist sequences \(\{T_i\}\) and \(\{S_i\}\) such that
\[
0 < T_i < S_i < t_i < T_2 < S_2 < t_2 < \cdots < S_n < b
\]
and
\[
\int_{-A}^{\alpha_1} \frac{d\zeta}{f(\zeta)} = \int_0^{T_1} g(t) \, dt, \quad \int_{\alpha_i}^{C_i} \frac{d\zeta}{f(\zeta)} = \int_{S_i}^{t_i} g(t) \, dt,
\]
\[
\int_{C_i}^{\alpha_{i+1}} \frac{d\zeta}{f(\zeta)} = \int_{t_i}^{T_{i+1}} g(t) \, dt \quad (i = 1, \ldots, n-1),
\]
\[
\int_{\alpha_n}^{B} \frac{d\zeta}{f(\zeta)} = \int_{S_n}^{b} g(t) \, dt.
\]

The following theorem establishes conditions under which the two-point boundary value problem has a unique solution. Extension to the multi-point problem is straightforward and will be omitted.

**Theorem 2.** Let the hypotheses of Theorem 1 hold and, in addition, suppose that \(f\) vanishes on \([-A, B]\) only at zero and that \(f\) is locally Lipschitz on \([-A, 0) \cup (0, B]\). Then the solution of the two-point boundary value problem (1) is unique.

**Proof.** Since \(f\) is locally Lipschitz, the standard existence-uniqueness theorem forces uniqueness of the solutions \(\dot{y}_1\) and \(\dot{y}_2\) constructed in the proof of Theorem 1; \(T_1\) and \(T_2\) are also unique. Since any solution of the differential equation is nondecreasing, the solution \(y(t)\) of the boundary value problem given by (6) is now seen to be unique.

**Example.** Let \(g(t) \equiv 1, f(y) \equiv |y|^{\alpha}\). Then (1) has no solution unless \(0 < \alpha < 1\). For \(0 < \alpha < 1\), the two-point boundary value problem (1) has a solution, which is unique, if and only if \((1 - \alpha)b \geq A^{1-\alpha} + B^{1-\alpha}\).
We now turn to existence of solutions of the two-point boundary value problem for \( y' = h(t, y) \); this will be established by means of a comparison theorem. A similar result for the multi-point problem is easily established along the same lines.

The following lemma is well known; a proof may be found in [4].

**Lemma.** Let \( g \) be continuous on \( E \), an open set in \( \mathbb{R}^2 \), and let the maximal solution \( u(t) \) of

\[
(7) \quad u' = g(t, u), \quad u(t_0) = u_0
\]

exist on \([t_0, t_0 + a)\). Let \( v(t) \) be continuous on \([t_0, t_0 + a)\) with \((t, v(t)) \in E\) for \( t \in [t_0, t_0 + a)\), and suppose that

\[
v'(t) < g(t, v(t)), \quad v(t_0) < u_0 \quad (t_0 < t < t_0 + a).
\]

Then

\[
v(t) < u(t) \quad \text{for} \quad t_0 < t < t_0 + a.
\]

Let the minimal solution \( w(t) \) of (7) exist on \((t_0 - a, t_0]\). Let \( v(t) \) be continuous on \((t_0 - a, t_0)\) with \((t, v(t)) \in E\) for \( t \in (t_0 - a, t_0]\), and suppose that

\[
v'(t) > g(t, v(t)), \quad v(t_0) > u_0 \quad (t_0 - a < t < t_0).
\]

Then

\[
v(t) > w(t) \quad \text{for} \quad t_0 - a < t < t_0.
\]

With the aid of this result, we can easily prove the following comparison theorem.

**Theorem 3.** Let the two-point boundary value problem

\[
z' = h(t, z), \quad z(0) = -\bar{A}, \quad z(b) = \bar{B},
\]

where \( \bar{A} \geq 0, \bar{B} \geq 0 \), possess a solution \( z(t) \), not necessarily unique. Suppose that

(i) \( 0 < A < \bar{A}, 0 < B < \bar{B} \),
(ii) \( h \) is continuous on an open set \( E \) containing \([0, b] \times [-\bar{A}, \bar{B}]\),
(iii) \( h(t, 0) \equiv 0 \) for \( t \in [0, b] \),
(iv) \( h(t, u) \geq \bar{h}(t, u) \) on \( E \).

Then the boundary value problem

\[
y' = h(t, y), \quad y(0) = -A, \quad y(b) = B
\]

has a solution.

**Proof.** Let \( y_1 \) be the maximal solution of the initial value problem

\[
y' = h(t, y), \quad y(0) = -A.
\]

Since \( z \) solves

\[
z(0) = -\bar{A} \leq -A, \quad z'(t) = \bar{h}(t, z(t)) \leq h(t, z(t)),
\]

we have from the first part of the Lemma that \( z(t) \leq y_1(t) \) as far to the right of zero as \( y_1 \) exists. Let

\[
t_1 = \min\{t \in [0, b] : z(t) = 0\}.
\]

Since the maximal solution \( y_1 \) can be extended until it leaves the set \([0, b] \times [-A, B]\), there must exist \( T_1 \leq t_1 \) such that \( y_1(T_1) = 0 \). From the differential equation (8) we get that \( y'_1(T_1 -) = 0 \).
Let \( y_2(t) \) be the minimal solution of the terminal value problem

\[ y'_2 = h(t, y_2), \quad y_2(b) = B \]

and let

\[ t_2 = \max\{t \in [0, b]: z(t) = 0\}. \]

Necessarily \( t_2 \geq t_1 \). By the second part of the Lemma we have that \( y_2(t) \leq z(t) \) as far to the left of \( b \) as \( y_2 \) exists; as before it follows that there is a \( T_2 \geq t_2 \) such that \( y_2 \) is defined on \([T_2, b]\) and \( y_2(T_2) = y'_2(T_2+) = 0\). Since \( T_2 \geq T_1 \), it follows that

\[
y(t) = \begin{cases} 
y_1(t), & 0 < t < T_1, \\
0, & T_1 \leq t \leq T_2, \\
y_2(t), & T_2 < t < b,
\end{cases}
\]

is a classical solution of the two-point boundary value problem (8).

This comparison theorem can be used both to prove existence and to prove nonexistence of solutions of the two-point boundary value problem. As an example of the former, the following result is immediate.

**COROLLARY.** Let \( f, g, A, B, b \) satisfy the hypotheses of Theorem 1, and let

\[ h(t, y) \geq g(t)f(y), \]

where \( h \) also satisfies hypotheses (ii)–(iii) of Theorem 3. Then the boundary value problem (8) has at least one solution.

**THEOREM 4.** Let \( A, B > 0; \) let \( h(t, y) > 0 \) for \((t, y) \in [0, b] \times [-A, B]\) and vanish precisely when \( y = 0\); let \( h \) be continuous on \([0, b] \times [-A, B]\) and locally Lipschitz on \([0, b] \times ([-A, 0) \cup (0, B])\). Then the two-point boundary value problem

\[ y' = h(t, y), \quad y(0) = -A, \quad y(b) = B \]

cannot have two distinct solutions.

The proof does not differ materially from that of Theorem 2 and so will be omitted, as will the extension to multi-point problems.

**REMARK.** The simple technique employed here can be applied readily to other forms of boundary conditions; for example, to the integral conditions imposed in [5].

**REFERENCES**


