PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

BOUNDARY VALUE PROBLEMS FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Conditions sufficient to guarantee existence and uniqueness of solutions to multipoint boundary value problems for the first-order differential equation $y' = h(t,y)$ are given when h fails to be Lipschitz along a solution of $y' = h(t,y)$ and the initial-value problem thus has nonunique solutions.

It is well known that the initial value problem for the first-order differential equation $y' = h(t,y)$ does not generally have a unique solution if h fails to be Lipschitz in y. This raises the possibility, for non-Lipschitz h, of well-posedness of problems that would be overspecified if h satisfied a Lipschitz condition; in particular, of the reasonableness of problems that would normally be associated with higher-order equations [**1**, **3**]. Here we examine existence and uniqueness of solutions to two- and multi-point boundary value problems when $y' = h(t,y)$ has a solution $y = a(t)$ along which h fails to be Lipschitz. Making the change of variable $y - a(t) \to y$, we may without loss of generality assume that h vanishes when $y = 0$ and that h is not Lipschitz in any neighborhood of $y = 0$. We treat first the case of separable variables $h(t,y) = g(t)f(y)$ and then use a comparison theorem to treat the more general case.

Let $b > 0$ and consider the boundary value problem

$$y' = g(t)f(y), \quad y(0) = -A, \quad y(b) = B \tag{1}$$

when $A > 0, B > 0, g > 0$, and $yf(y) < 0$ for $y \neq 0$. Then $y' < 0$ for $y > 0$ and $y' > 0$ for $y < 0$, from which it is clear that the problem has no solution. Similarly there is no solution if $yf(y) > 0$ for $y \neq 0$. Thus if solutions are to exist in general, then the right-hand side cannot change sign.

It is interesting to note that, since the two-point boundary value problem

$$y'' = \lambda y^{1/2}, \qquad y(0) = y(b) = A > 0$$

has a unique nonnegative solution for each $b > 0$ and this solution vanishes on an interval when λ is sufficiently large [**2**], the following considerations do not extend to higher-order equations.

THEOREM 1. *Let $A, B \geq 0$ and let $f \geq 0$ be continuous on $[-A, B]$ and vanish on $(-A, B)$ precisely at $\alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} = 0 < \cdots < \alpha_n$. Let $g > 0$ be continuous on $[0, b]$. Then there is no solution of the two-point boundary value problem* (1) *unless the improper integral*

$$\int_{-A}^{B} \frac{d\varsigma}{f(\varsigma)} \tag{2}$$

Received by the editors January 23, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34B15, 34B10.

exists. If this integral converges, (1) *has continuously differentiable solutions if and only if there exist* T_1 *and* T_2 *in* $[0,b]$ *satisfying*

$$\int_0^{T_1} g(s)\,ds = \int_{-A}^{0} \frac{d\varsigma}{f(\varsigma)}, \qquad \int_{T_2}^{b} g(s)\,ds = \int_0^{B} \frac{d\varsigma}{f(\varsigma)}, \tag{3}$$

and

$$T_1 \le T_2. \tag{4}$$

PROOF. We show first the necessity of these conditions. Suppose $y(t)$ is a solution of (1) and set $\alpha_0 = -A$, $\alpha_{n+1} = B$ (f may or may not vanish at these points). On each (α_i, α_{i+1}), $f \neq 0$. Define

$$t_i = \sup\{t\colon y(t) < \alpha_i\}, \qquad i = 1, 2, \ldots, n+1,$$
$$s_i = \inf\{t\colon y(t) > \alpha_i\}, \qquad i = 0, 1, \ldots, n.$$

Then $f(y(t)) \neq 0$ on (s_i, t_{i+1}). Let $\varepsilon, \delta > 0$ be sufficiently small. Dividing the differential equation by $f(y(t))$, integrating from $s_i + \varepsilon$ to $t_{i+1} - \delta$, and passing to the limit as $\varepsilon, \delta \to 0$ yields

$$\int_{\alpha_i}^{\alpha_{i+1}} \frac{d\varsigma}{f(\varsigma)} = \int_{y(s_i)}^{y(t_{i+1})} \frac{y'(t)\,dt}{f(y(t))} = \int_{s_i}^{t_{i+1}} g(t)\,dt, \qquad i = 0, \ldots, n.$$

Thus the improper integral (2) converges. It also follows that

$$\int_{-A}^{0} \frac{d\varsigma}{f(\varsigma)} = \sum_{i=0}^{k} \int_{s_i}^{t_{i+1}} g(t)\,dt \le \int_0^{t_{k+1}} g(t)\,dt;$$

hence there exists a $T_1 \le t_{k+1}$ such that the first equality of (3) holds. In the same way, there exists $T_2 \ge t_{k+1}$ such that the second equality of (3) obtains. The necessity of the hypotheses follows.

To show sufficiency, let $\hat{y}$ be the maximal solution of the initial value problem $y' = g(t)f(y)$, $y(0) = -A$; we shall show that $\hat{y}(T_1) = 0$. Let α_i, $i = 0, \ldots, n+1$, be as before; then convergence of the improper integral (2) guarantees the existence of the improper integrals

$$\int_{\alpha_i}^{\alpha_{i+1}} \frac{d\varsigma}{f(\varsigma)}, \qquad i = 0, \ldots, n.$$

Since T_1 satisfying (3) exists, there also exist $t_1 < t_2 < \cdots < t_{k+1} \le T_1$ such that

$$\sum_{i=1}^{j} \int_{\alpha_{i-1}}^{\alpha_i} \frac{d\varsigma}{f(\varsigma)} = \int_0^{t_j} g(s)\,ds \qquad (j = 1, \ldots, k+1);$$

set $t_0 = 0$.

Let y_i ($i = 0, \ldots, k$) be the maximal solution of the initial value problem

$$y_i'(t) = g(t)f(y_i(t)), \qquad y_i(t_i) = \alpha_i;$$

we shall show first that y_i is defined on $[t_i, t_{i+1}]$ and $y_i(t_{i+1}) = \alpha_{i+1}$. To this end let $y_{i,\varepsilon}$, for all sufficiently small $\varepsilon > 0$, be the maximal solution of

$$y_{i,\varepsilon}' = g(t)f(y_{i,\varepsilon}) + \varepsilon, \qquad y_{i,\varepsilon}(t_i) = \alpha_i + \varepsilon. \tag{5}$$

If y_i is defined on $[t_i, s_i)$, then for any $\delta > 0$, $y_{i,\varepsilon}$ exists for $t_i \leq t \leq s_i - \delta$ for all sufficiently small ε, and $y_{i,\varepsilon} \downarrow y_i$ as $\varepsilon \downarrow 0$ [**4**]. Suppose that $y_i(t) \equiv \alpha_i$ on $[t_i, \bar{t}_i]$ with $\bar{t}_i > t_i$ and $\bar{t}_i$ small enough that θ defined by

$$\int_{\alpha_i}^{\theta} \frac{d\varsigma}{f(\varsigma)} = \int_{\bar{t}}^{\bar{t}_i} g(t)\, dt$$

exists and satisfies $\theta < \alpha_{i+1}$; this is guaranteed by the existence of (2). Since $g > 0$, $\theta > \alpha_i$. For $\varepsilon_0 > 0$ sufficiently small, the maximal solution $y_{i,\varepsilon}$ of (5) exists on $[t_i, \bar{t}_i]$ for $0 < \varepsilon \leq \varepsilon_0$. By further reducing $\bar{t}_i$ and θ, if necessary, we may assume that $y_{i,\varepsilon}(t) < \alpha_{i+1}$ for $0 < \varepsilon \leq \varepsilon_0$ and $t \in [t_i, \bar{t}_i]$. Then $\alpha_i < y_{i,\varepsilon}(t) < \alpha_{i+1}$ on $(t_i, \bar{t}_i)$, so $f(y_{i,\varepsilon}(t)) > 0$ there, and we get from (5) that

$$\int_{t_i}^{\bar{t}_i} \frac{y'_{i,\varepsilon}(t)\, dt}{f(y_{i,\varepsilon}(t))}\, dt > \int_{t_i}^{\bar{t}_i} g(t)\, dt$$

and hence that

$$\int_{\alpha_i}^{y_{i,\varepsilon}(\bar{t}_i)} \frac{d\varsigma}{f(\varsigma)} > \int_{\alpha_i+\varepsilon}^{y_{i,\varepsilon}(\bar{t}_i)} \frac{d\varsigma}{f(\varsigma)} > \int_{t_i}^{\bar{t}_i} g(t)\, dt.$$

Thus $y_{i,\varepsilon}(\bar{t}_i) > \theta$ for $0 < \varepsilon \leq \varepsilon_0$, whence $y_i(\bar{t}_i) = \lim_{\varepsilon \to 0} y_{i,\varepsilon}(\bar{t}_i) \geq \theta > \alpha_i$, a contradiction. We have therefore shown that $y_i(t) = \alpha_i$ only for $t = t_i$.

Suppose now that $\alpha_i < y_i(t) < \alpha_{i+1}$ for $t_i < t < s$. Then $f(y_i(t)) > 0$, so we get that

$$\int_{y_i(t_i+\varepsilon)}^{y_i(s-\delta)} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i+\varepsilon}^{s-\delta} g(t)\, dt$$

for sufficiently small $\varepsilon, \delta > 0$. Passing to the limit as $\varepsilon \to 0$, $\delta \to 0$, we have that

$$\int_{\alpha_i}^{y_i(s-)} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{s} g(t)\, dt.$$

But we have

$$\int_{\alpha_i}^{\alpha_{i+1}} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{t_{i+1}} g(t)\, dt.$$

Therefore if $s < t_{i+1}$, then $y_i(s-) < \alpha_{i+1}$ and y_i can be extended to the right of s. So $s \geq t_{i+1}$. But the unique solution to

$$\int_{\alpha_i}^{\theta} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{t_{i+1}} g(s)\, ds$$

is $\theta = \alpha_{i+1}$. Therefore $y_i(t_{i+1}) = \alpha_{i+1}$.

We have shown that on each interval $[\alpha_i, \alpha_{i+1}]$ the maximal solution y_i of $y'_i = g(t)f(y_i)$, $y_i(t_i) = \alpha_i$ exists and satisfies $y_i(t_{i+1}) = \alpha_{i+1}$. Since $f(\alpha_i) = 0$ for $i = 1, \ldots, k$, we have easily that $y'_i(\alpha_i-) = y'_i(\alpha_{i+1}+) = 0$ $(i = 1, \ldots, k)$ and $y'_0(\alpha_1-) = 0$. It follows that $\hat{y}_1$ defined by

$$\hat{y}_1(t) = y_i(t), \qquad t_i \leq t < t_{i+1},$$

is the maximal solution of $y' = g(t)f(y)$, $y(0) = -A$ on $[0, T_1)$ and $\hat{y}_1(T_1-) = \hat{y}'_1(T_1-) = 0$.

In the same way we conclude there exists the minimal solution of

$$\hat{y}_2'(t) = g(t)f(\hat{y}_2(t)), \qquad \hat{y}_2(b) = B$$

defined on $(T_2, b]$ and satisfying $\hat{y}_2(T_2+) = \hat{y}_2'(T_2+) = 0$. Therefore, since $T_1 \leq T_2$,

$$y(t) \equiv \begin{cases} \hat{y}_1(t), & 0 \leq t < T_1, \\ 0, & T_1 \leq t \leq T_2, \\ \hat{y}_2(t), & T_2 < t \leq b, \end{cases} \tag{6}$$

defines a classical solution of (1), proving the theorem.

COROLLARY 1. *If f and g are as in the theorem and $\int_0^\infty g = \int_{-\infty}^0 g = \infty$, then the two-point boundary problem* (1) *has a solution for each $A, B \geq 0$ provided b is sufficiently large* (*depending on A and B*).

The following result is an easy extension to the multi-point boundary value problem. It is interesting to compare it with corresponding results for higher-order Lipschitz equations [**3**].

COROLLARY 2. *Let f and g satisfy the hypotheses of the theorem. Let $C_i \in (\alpha_i, \alpha_{i+1})$ for $i = 1, \ldots, n-1$ and $0 < t_1 < \cdots < t_{n-1} < b$ be given. Then the boundary value problem*

$$y' = g(t)f(y), \quad y(0) = -A, \quad y(t_1) = C_1, \ldots, y(t_{n-1}) = C_{n-1}, \quad y(b) = B$$

has a solution if and only if there exist sequences $\{T_i\}$ and $\{S_i\}$ such that

$$0 < T_1 \leq S_1 < t_1 < T_2 \leq S_2 < t_2 < \cdots \leq S_n < b$$

and

$$\int_{-A}^{\alpha_1} \frac{d\varsigma}{f(\varsigma)} = \int_0^{T_1} g(t)\,dt, \qquad \int_{\alpha_i}^{C_i} \frac{d\varsigma}{f(\varsigma)} = \int_{S_i}^{t_i} g(t)\,dt,$$

$$\int_{C_i}^{\alpha_{i+1}} \frac{d\varsigma}{f(\varsigma)} = \int_{t_i}^{T_{i+1}} g(t)\,dt \qquad (i = 1, \ldots, n-1),$$

$$\int_{\alpha_n}^{B} \frac{d\varsigma}{f(\varsigma)} = \int_{S_n}^{b} g(t)\,dt.$$

The following theorem establishes conditions under which the two-point boundary value problem has a unique solution. Extension to the multi-point problem is straightforward and will be omitted.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold and, in addition, suppose that f vanishes on $[-A, B]$ only at zero and that f is locally Lipschitz on $[-A, 0) \cup (0, B]$. Then the solution of the two-point boundary value problem* (1) *is unique.*

PROOF. Since f is locally Lipschitz, the standard existence-uniqueness theorem forces uniqueness of the solutions $\hat{y}_1$ and $\hat{y}_2$ constructed in the proof of Theorem 1; T_1 and T_2 are also unique. Since any solution of the differential equation is nondecreasing, the solution $y(t)$ of the boundary value problem given by (6) is now seen to be unique.

EXAMPLE. Let $g(t) \equiv 1$, $f(y) \equiv |y|^\alpha$. Then (1) has no solution unless $0 < \alpha < 1$. For $0 < \alpha < 1$, the two-point boundary value problem (1) has a solution, which is unique, if and only if $(1-\alpha)b \geq A^{1-\alpha} + B^{1-\alpha}$.

We now turn to existence of solutions of the two-point boundary value problem for $y' = h(t, y)$; this will be established by means of a comparison theorem. A similar result for the multi-point problem is easily established along the same lines.

The following lemma is well known; a proof may be found in [**4**].

LEMMA. *Let g be continuous on E, an open set in $\mathbf{R}^2$, and let the maximal solution $u(t)$ of*

$$u' = g(t,u), \qquad u(t_0) = u_0 \tag{7}$$

exist on $[t_0, t_0 + a)$. Let $v(t)$ be continuous on $[t_0, t_0 + a)$ with $(t, v(t)) \in E$ for $t \in [t_0, t_0 + a)$, and suppose that

$$v'(t) \leq g(t, v(t)), \quad v(t_0) \leq u_0 \qquad (t_0 < t < t_0 + a).$$

Then

$$v(t) \leq u(t) \quad \text{for } t_0 \leq t < t + a.$$

Let the minimal solution $w(t)$ of (7) exist on $(t_0 - a, t_0]$. Let $v(t)$ be continuous on $(t_0 - a, t_0]$ with $(t, v(t)) \in E$ for $t \in (t_0 - a, t_0]$, and suppose that

$$v'(t) \leq g(t, v(t)), \quad v(t_0) \geq u_0 \qquad (t_0 - a < t < t_0).$$

Then

$$v(t) \geq w(t) \quad \text{for } t_0 - a < t \leq t_0.$$

With the aid of this result, we can easily prove the following comparison theorem.

THEOREM 3. *Let the two-point boundary value problem*

$$z' = \tilde{h}(t,z), \quad z(0) = -\tilde{A}, \quad z(b) = \tilde{B},$$

where $\tilde{A} \geq 0, \tilde{B} \geq 0$, possess a solution $z(t)$, not necessarily unique. Suppose that

(i) $0 \leq A \leq \tilde{A}$, $0 \leq B \leq \tilde{B}$,

(ii) *h is continuous on an open set E containing* $[0,b] \times [-\tilde{A}, \tilde{B}]$,

(iii) $h(t,0) \equiv 0$ *for* $t \in [0,b]$,

(iv) $h(t,u) \geq \tilde{h}(t,u)$ *on* E.

Then the boundary value problem

$$y' = h(t,y), \quad y(0) = -A, \quad y(b) = B \tag{8}$$

has a solution.

PROOF. Let y_1 be the maximal solution of the initial value problem

$$y' = h(t,y), \qquad y(0) = -A.$$

Since z solves

$$z(0) = -\tilde{A} \leq -A, \qquad z'(t) = \tilde{h}(t, z(t)) \leq h(t, z(t)),$$

we have from the first part of the Lemma that $z(t) \leq y_1(t)$ as far to the right of zero as y_1 exists. Let

$$t_1 = \min\{t \in [0,b] \colon z(t) = 0\}.$$

Since the maximal solution y_1 can be extended until it leaves the set $[0,b] \times [-A, B]$, there must exist $T_1 \leq t_1$ such that $y_1(T_1) = 0$. From the differential equation (8) we get that $y_1'(T_1-) = 0$.

Let $y_2(t)$ be the minimal solution of the terminal value problem

$$y_2' = h(t, y_2), \qquad y_2(b) = B$$

and let

$$t_2 = \max\{t \in [0,b] \colon z(t) = 0\}.$$

Necessarily $t_2 \geq t_1$. By the second part of the Lemma we have that $y_2(t) \leq z(t)$ as far to the left of b as y_2 exists; as before it follows that there is a $T_2 \geq t_2$ such that y_2 is defined on $[T_2, b]$ and $y_2(T_2) = y_2'(T_2+) = 0$. Since $T_2 \geq T_1$, it follows that

$$y(t) \equiv \begin{cases} y_1(t), & 0 \leq t < T_1, \\ 0, & T_1 \leq t \leq T_2, \\ y_2(t), & T_2 < t \leq b, \end{cases}$$

is a classical solution of the two-point boundary value problem (8).

This comparison theorem can be used both to prove existence and to prove nonexistence of solutions of the two-point boundary value problem. As an example of the former, the following result is immediate.

COROLLARY. *Let f, g, A, B, b satisfy the hypotheses of Theorem* 1, *and let*

$$h(t, y) \geq g(t) f(y),$$

where h also satisfies hypotheses (ii)–(iii) *of Theorem* 3. *Then the boundary value problem* (8) *has at least one solution.*

THEOREM 4. *Let $A, B > 0$; let $h(t,y) \geq 0$ for $(t,y) \in [0,b] \times [-A, B]$ and vanish precisely when $y = 0$; let h be continuous on $[0,b] \times [-A, B]$ and locally Lipschitz on $[0,b] \times ([-A, 0) \cup (0, B])$. Then the two-point boundary value problem*

$$y' = h(t,y), \quad y(0) = -A, \quad y(b) = B$$

cannot have two distinct solutions.

The proof does not differ materially from that of Theorem 2 and so will be omitted, as will the extension to multi-point problems.

REMARK. The simple technique employed here can be applied readily to other forms of boundary conditions; for example, to the integral conditions imposed in [**5**].

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