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THE EXISTENCE OF UNIVERSAL INVARIANT SEMIREGULAR MEASURES ON GROUPS

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ABSTRACT. A nonnegative, countably additive, extended real-valued measure is universal on a set X iff it is defined on all subsets of X, and is semiregular iff every set of positive measure contains a subset of positive finite measure. We prove that on every group of sufficiently large cardinality there exists a universal invariant semiregular measure vanishing on singletons. Thus we give complete solutions to the problems stated by Kannan and Raju [**4**] and Pelc [**5**].

In this paper we study universal semiregular invariant measures on groups.

By a universal measure on a set X we mean a function m defined on a σ-algebra $P(X)$ of all subsets of X such that:

(1) the values of m are nonnegative reals or $+\infty$,

(2) m is countably additive,

(3) $m(\{x\}) = 0$ for any $x \in X$,

(4) $m(X) > 0$.

A universal measure on X is:

—finite iff $m(X) < +\infty$,

—σ-finite iff X is a countable union of sets of finite measure,

—semiregular iff every set of positive measure contains a subset of positive finite measure.

The smallest cardinal κ (if it exists) such that there exists a set of power κ with a universal semiregular measure will be denoted by κ_0. It is easy to observe that on a set X of power κ_0 there exists also a universal finite measure and any such measure is uniform, i.e. $m(A) = 0$ for every $A \subset X$, $|A| < |X|$. It is well known that the existence of the cardinal κ_0 is unprovable in usual set theory.

A universal measure m on a group $(G, \cdot)$ is invariant iff $m(g \cdot A) = m(A)$ for every $g \in G$, $A \subset G$. The unit element of a group G will be denoted by e.

Erdös and Mauldin [**3**] proved that there are no universal invariant, σ-finite measures on any group. Kannan and Raju [**4**] asked whether it is possible for a group G to carry a universal, invariant, semiregular measure. It is obvious that positive answer implies that $|G| \geq \kappa_0$. Under this assumption A. Pelc [**5**] gives a positive solution for abelian groups. In the present paper we extend the result of Pelc to arbitrary groups of power $\geq \kappa_0$.

The main idea of our construction is given in the following

LEMMA. *If a group G contains a subset S such that:*

(i) $|S| = \kappa_0$,

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(ii) $|S \cap g \cdot S| < \kappa_0$ *for every* $g \in G\backslash\{e\}$,
then there exists a universal, semiregular, invariant measure on G.

PROOF. It follows from the definition of the cardinal κ_0 that we can find a finite universal measure m on S. We define a universal measure m_1 on G by

$$m_1(A) = \sum_{g \in G} m(g \cdot A \cap S) \quad \text{for every } A \subset G.$$

It is easy to see that m_1 is in fact a universal, invariant measure on G.

We check that m_1 is semiregular. If $m_1(A) > 0$, then there is some $g_0 \in G$ such that $0 < m(g_0 \cdot A \cap S) < +\infty$. Then $A \cap g_0^{-1} \cdot S \subset A$ and it is enough to show that $m_1(A \cap g_0^{-1} \cdot S) = m(g_0 \cdot A \cap S)$. By condition (ii), for any $g \in G$ such that $g \neq g_0$ we have $|g \cdot A \cap g \cdot g_0^{-1} \cdot S \cap S| < \kappa_0$. Hence in view of the uniformity of m we get $m(g \cdot A \cap g \cdot g_0^{-1} \cdot S \cap S) = 0$. Finally

$$m_1(A \cap g_0^{-1} \cdot S) = \sum_{g \in G} m(g \cdot A \cap g \cdot g_0^{-1} \cdot S \cap S) = m(g_0 \cdot A \cap S). \quad \square$$

Now we can formulate our main result.

THEOREM. *Let* $(G, \cdot)$ *be an arbitrary group such that* $|G| \geq \kappa_0$. *Then there exists a universal, invariant, semiregular measure on* G.

PROOF. Let us fix a subgroup H of G such that $|H| = \kappa_0$.

It is easy to construct by induction an increasing sequence $\langle H_\alpha \colon \alpha < \kappa_0 \rangle$ of subgroups of H with the following properties (cf. Hulanicki [**2**]):

(i) $\forall \alpha, \beta \ \alpha < \beta \rightarrow H_\alpha \subset H_\beta$,
(ii) $\bigcup_{\alpha < \kappa_0} H_\alpha = H$,
(iii) $\forall \alpha |H_\alpha| < \kappa_0$,
(iv) $\forall \alpha \ Q_\alpha = H_\alpha \backslash \bigcup_{\beta < \alpha} H_\beta \neq \emptyset$.

From (i)–(iv) it easily follows that $\forall \alpha \forall g \in H_\alpha \forall \beta > \alpha (g \cdot Q_\beta = Q_\beta)$.

Let S be a selector of the family $\{Q_\alpha \colon \alpha < \kappa_0\}$. We show that S satisfies the hypotheses of the Lemma. Obviously $|S| = \kappa_0$. Let $g \in G$, $g \neq e$ and consider two cases:

Case I. $g \notin H$. Then $g \cdot S \cap S = \emptyset$ since $S \subset H$ and $g \cdot H \cap H = \emptyset$.

Case II. $g \in H$. Then there exists $\alpha < \kappa_0$ such that $g \in H_\alpha$.

We claim that $g \cdot S \cap S \subset H_\alpha$. Suppose conversely that there exist $\beta > \alpha$ and $s \in S$ such that $g \cdot s \in S \cap Q_\beta$. Since $g \cdot Q_\beta = Q_\beta$ we get $s \in S \cap Q_\beta$, hence $g \cdot s = s$. This contradicts the choice of g. Now by (iii) we have $|g \cdot S \cap S| < \kappa_0$. Q.E.D.

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