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SYMMETRIZATION AND OPTIMAL CONTROL FOR ELLIPTIC EQUATIONS

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ABSTRACT. We consider an optimal control problem where $u(x)$ satisfies $-\operatorname{div}(H(x)\nabla u) = 1$ in Ω and $H(x)$ is a control. We introduce the functional $J_\Omega(H) = |\Omega|^{-1} \int_\Omega u(x)\,dx$ and show using a symmetrization argument that if the distribution function of H is fixed, then $J_\Omega(H)$ is largest when Ω is a ball and H is radial and decreasing on radii.

0. Introduction. In [**4**], Murat and Tartar considered the following problem: Let B be a ball in $\mathbf{R}^n$ and let $u(x)$ satisfy

$$-\operatorname{div}(H(x)\nabla u) = 1 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{0.1}$$

where $H(x)$ is a control variable in a class $\mathcal{H}$,

$$\mathcal{H} = \left\{ H \in L^\infty(B)\colon \alpha \le H(x) \le \beta,\ \int_B H\,dx = \gamma \right\},$$

and α, β, γ are fixed positive constants. Determine the control $H_0(x)$ which maximizes

$$J_B(H) = \frac{1}{|B|} \int_B u(x)\,dx$$

over the class $\mathcal{H}$. They find that $H_0(x) = \beta\chi_{B_\rho} + \alpha\chi_{B/B_\rho}$, where B_ρ is a ball of radius ρ, concentric with B, and χ_E is the characteristic function of the set E.

In this paper we extend the class of controls $\mathcal{H}$ to be

$$\mathcal{H}_1 = \left\{ H \in L^\infty(B),\ \alpha(x) \le H(x) \le \beta(x),\ \int_B H\,dx = \gamma \right\}, \tag{0.2}$$

where α and β are fixed bounded functions which are radial, decreasing on radii, and satisfy $0 < \alpha_0 \le \alpha(x) \le \beta(x)$ a.e. in B. We find that the unique optimal control has the form

$$H_0(x) = \beta(x)\chi_{B_\rho} + \alpha(x)\chi_{B/B_\rho}.$$

While Murat and Tartar deduce their result as an application of homogenization, we employ a comparison theorem (Theorem 1.1 below) for solutions of (0.1) involving rearrangement of the control H. This result enables us to reduce $\mathcal{H}_1$ to consist of radial functions, in which case the characterization of the optimal control is straightforward.

The comparison theorem has a long history dating back to 1856 when Saint Venant [**6**] conjectured the result in the case that H is constant. A proof of the theorem in this case was obtained by Pólya in 1948 [**5**].

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1. Main results. If H is a positive function in $L^\infty(\Omega)$, $H \geq \alpha > 0$, equation (0.1) has a unique weak solution in the Sobolev space $H_0^1(\Omega)$. This solution u is characterized by the identity

$$\int_\Omega H\nabla u \cdot \nabla \varsigma \, dx = \int_\Omega \varsigma \, dx \quad \forall \varsigma \in C_0^1(\Omega). \tag{1.1}$$

It is known [**2**] that the solution u is actually locally Hölder continuous.

If $u \in H_0^1(\Omega)$ is the weak solution of (0.1), we define

$$J_\Omega(H) = \frac{1}{|\Omega|}\int_\Omega u \, dx, \tag{1.2}$$

where $|\Omega|$ denotes the measure of Ω. We will frequently drop the domain dependence in denoting J if there is no danger of ambiguity.

Suppose Ω is a bounded domain and let f be a positive measurable function on Ω. We denote the decreasing rearrangement of f by f^*. Further, if B is the ball centered at the origin such that $|B| = |\Omega|$, the symmetric decreasing rearrangement of f is the function $f^\#$ defined on B by the formula

$$f^\#(x) = f^*(C_n|x|^n), \qquad x \in B,$$

where $C_n = \pi^{n/2}/\Gamma(1+n/2)$ is the measure of the unit ball in $\mathbf{R}^n$. We shall also need the following inequality of Hardy and Littlewood [**3**]. If f and g re nonnegative on Ω, then

$$\int_\Omega fg \, dx \leq \int_0^{|\Omega|} f^*(s)g^*(s) \, ds. \tag{1.3}$$

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ $(n \geq 1)$ and H be a measurable function on Ω so $\alpha \leq H \leq \beta$ for positive constants α and β. If $H^\#$ is the symmetric decreasing rearrangement of H on a ball $B \subset \mathbf{R}^n$, then*

$$J_\Omega(H) \leq J_B(H^\#). \tag{1.4}$$

Our proof of Theorem 1.1 is modelled after the argument of Talenti [**8**]. We note, however, that since our functional J is lower semicontinuous [**1**], we can avoid the geometric measure theory in [**8**] and reduce to the case $H \in C^\infty(\Omega)$.

PROOF OF THEOREM 1.1. If $n = 1$, the theorem is easily proven so we take $n > 1$. Let R denote the radius of B, so $C_nR^n = |B| = |\Omega|$. Let $u \in H_0^1(\Omega)$ and $v \in H_0^1(B)$ be the weak solutions of

$$-\operatorname{div}(H\nabla u) = 1 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.5}$$

$$-\operatorname{div}(H^\#\nabla v) = 1 \quad \text{in } B, \qquad v = 0 \text{ on } \partial B. \tag{1.6}$$

Note that since $H^\#$ is radial, v is radial and we have

$$\int_B v \, dx = \frac{C_n}{n}\int_0^R \frac{r^{n+1}}{H^\#(r)} \, dr = \frac{1}{n^2C_n^{2/n}}\int_0^{|\Omega|} \frac{s^{2/n}}{H^*(s)} \, ds. \tag{1.7}$$

To obtain the estimate (1.4), we assume at first that $H \in C^\infty(\Omega)$. This restriction will be removed later by a passage to the limit. Since $H \in C^\infty(\Omega)$, $u \in C^\infty(\Omega)$ (see [**2**, p. 109]). By the maximum principle, $u > 0$ in Ω, and if $\mu(\lambda) = |\{u > \lambda\}|$ is the distibution function of u, then μ is strictly decreasing, so $u^* = \mu^{-1}$ on the range of μ. If $\lambda > 0$ is not a critical value of u, then the open set $\{u > \lambda\}$ is bounded by the smooth compact manifold $\{u = \lambda\}$ which is contained in Ω. We denote by $d\sigma$ the surface element on this surface. Note that by Sard's Theorem [**7**], the set of critical values of u has measure zero.

If $\lambda > 0$ is not a critical value of u, we may integrate both sides of (1.5) over the set $\{u > \lambda\}$ and apply the divergence theorem to obtain

$$\mu(\lambda) = \int_{u>\lambda} dx = -\int_{u>\lambda} \operatorname{div}(H\nabla u)\, dx = -\int_{u=\lambda} H\nabla u \cdot n\, d\sigma,$$

where n is the outward unit normal to $\{u = \lambda\}$. Since $n = -\nabla u/|\nabla u|$, we see that

$$\mu(\lambda) = \int_{u=\lambda} H|\nabla u|\, d\sigma. \tag{1.8}$$

We now apply the isoperimetric inequality to estimate the surface area of $\{u = \lambda\}$. Since this set bounds a region of measure $\mu(\lambda)$, we find that

$$\int_{u=\lambda} d\sigma \geq nC_n^{1/n}\mu(\lambda)^{1-1/n}. \tag{1.9}$$

Note that the boundary condition (1.5) is essential to this estimate.

Using the Cauchy-Schwarz inequality, we get

$$\int_{u=\lambda} d\sigma \leq \left(\int_{u=\lambda} H|\nabla u|\, d\sigma\right)^{1/2} \left(\int_{u=\lambda} \frac{1}{H}\frac{1}{|\nabla u|}\, d\sigma\right)^{1/2}. \tag{1.10}$$

Inequalities (1.8), (1.9), and (1.10) yield

$$n^2C_n^{2/n}\mu(\lambda)^{1-2/n} \leq \int_{u=\lambda} \frac{1}{H}\frac{1}{|\nabla u|}\, d\sigma. \tag{1.11}$$

Integrating both sides of (1.11) between 0 and t, we obtain

$$n^2C_n^{2/n}\int_0^t \mu(\lambda)^{1-2/n}\, d\lambda \leq \int_0^t \int_{u=\lambda} \frac{1}{H}\frac{1}{|\nabla u|}\, d\sigma\, dt. \tag{1.12}$$

But $(1/|\nabla u|)\, d\sigma\, d\lambda$ is the volume element in Ω, so if we apply the Hardy-Littlewood inequality (1.3), we see that the right-hand side of (1.12) may be estimated as follows:

$$\begin{aligned}\int_0^t \int_{u=\lambda} \frac{1}{H}\frac{1}{|\nabla u|}\, d\sigma\, d\lambda = \int_{u<t} \frac{1}{H}\, dx &\leq \int_0^{|\Omega|} \left(\frac{1}{H}\right)^*(w)\chi^*_{\{u<t\}}(w)\, dw \\ &= \int_0^{|\Omega|-\mu(t)} \left(\frac{1}{H}\right)^*(w)\, dw.\end{aligned}$$

Thus

$$n^2C_n^{2/n}\int_0^t \mu(\lambda)^{1-2/n}\, d\lambda \leq \int_0^{|\Omega|-\mu(t)} \left(\frac{1}{H}\right)^*(w)\, dw. \tag{1.13}$$

Making the change of variables $s = \mu(t)$ in (1.13), we find that for $0 \leq s \leq |\Omega|$,

$$n^2C_n^{2/n}\int_0^{u^*(s)} \mu(\lambda)^{1-2/n}\, d\lambda \leq \int_0^{|\Omega|-s} \left(\frac{1}{H}\right)^*(w)\, dw. \tag{1.14}$$

Next we multiply both sides of (1.14) by $s^{2/n-1}$ and integrate from 0 to $|\Omega|$ to obtain

$$\begin{aligned} n^2 C_n^{2/n} &\int_0^{|\Omega|} s^{2/n-1} \int_0^{u^*(s)} \mu(\lambda)^{1-2/n}\, d\lambda\, ds \\ &\le \int_0^{|\Omega|} s^{2/n-1} \int_0^{|\Omega|-s} \left(\frac{1}{H}\right)^*(w)\, dw\, ds \\ &= \frac{n}{2} \int_0^{|\Omega|} s^{2/n} \left(\frac{1}{H}\right)^*(|\Omega|-s)\, ds \\ &= \frac{n}{2} \int_0^{|\Omega|} \frac{s^{2/n}}{H^*(s)}\, ds = \frac{n^3}{2} C_n^{2/n} \int_B v\, dx \quad \text{(by (1.7))}. \end{aligned}$$

Note that we have used $(1/H)^*(|\Omega|-s) = 1/H^*(s)$, since the two functions are increasing with the same distribution function.

Canceling constants in the preceding inequality and changing the order of integration yields

$$\begin{aligned} \int_B v\, dx &\ge \frac{2}{n} \int_0^{|\Omega|} s^{2/n-1} \int_0^{u^*(s)} \mu(\lambda)^{1-2/n}\, d\lambda\, ds \\ &= \frac{2}{n} \int_0^{u^*(0)} \int_0^{\mu(\lambda)} s^{2/n-1} \mu(\lambda)^{1-2/n}\, ds\, d\lambda \\ &= \int_0^{u^*(0)} \mu(\lambda)\, d\lambda = \int_0^{|\Omega|} u^*(s)\, ds = \int_\Omega u\, dx, \end{aligned}$$

which was to be shown.

To deduce the theorem for measurable H, we use the fact that the functional $H \mapsto J(H)$ is lower semicontinuous with respect to weak* convergence in $L^\infty(\Omega)$ (see [**1**, Theorem 2.1]). Let $H \in L^\infty(\Omega)$ and $a \le H \le b$ for positive constants a and b. Choose a sequence $H_m \in C^\infty(\Omega)$ so

(a) $H_m \to H$ in $L^2(\Omega)$ and a.e.,

(b) $a \le H_m \le b$ for all m.

Then $H_m \to H$ weak* in L^∞. Further, it follows from (1.3) that $H_m^* \to H^*$ in $L^2(0, |\Omega|)$, so, by passing to a subsequence if necessary, we may assume that

(c) $H_m^* \to H^*$ a.e. on $(0, |\Omega|)$.

Now let $u_m \in H_0^1(\Omega)$ and $v_m \in H_0^1(B)$ be the weak solutions to

$$-\operatorname{div}(H_m \nabla u_m) = 1 \quad \text{and} \quad -\operatorname{div}(H_m^\# \nabla v_m) = 1.$$

We obtain, using the lower semicontinuity of J and the dominated convergence theorem,

$$\begin{aligned} \int_\Omega u\, dx &\le \liminf_{m\to\infty} \int_\Omega u_m\, dx \\ &\le \liminf_{m\to\infty} \int_B v_m\, dx \quad (\text{since } H_m \in C^\infty(\Omega)) \\ &= \liminf_{m\to\infty} \frac{C_n^{-2/n}}{n^2} \int_0^{|\Omega|} \frac{s^{2/n}}{H_m^*(s)}\, ds \quad \text{(by (1.7))} \\ &= \frac{C_n^{-2/n}}{n^2} \int_0^{|\Omega|} \frac{s^{2/n}}{H^*(s)}\, ds = \int_B v\, dx. \end{aligned}$$

Thus $J_\Omega(H) \le J_B(H^\#)$ as was to be shown.

REMARK. If we take $s = 0$ in (1.14) we obtain

$$n^2 C_n^{2/n} \int_0^{u^*(0)} \mu(\lambda)^{1-2/n}\, d\lambda \le \int_0^{|\Omega|} \left(\frac{1}{H}\right)^* dx = \int_\Omega \frac{1}{H}\, dx.$$

This inequality is particularly simple if $n = 2$, in which case it beccomes

$$\|u\|_\infty = u^*(0) \le \frac{1}{4\pi} \int_\Omega \frac{1}{H}\, dx.$$

THEOREM 1.2. *Let B be a ball centered at the origin in $\mathbf{R}^n$ of radius R. Let α, β be radial functions in $L^\infty(B)$ which are decreasing on radii, and which satisfy* $\operatorname{ess\,inf} \alpha(x) > 0$ *and $\alpha(x) \le \beta(x)$ a.e. Let γ be a constant so*

$$\int_B \alpha\, dx \le \gamma \le \int_B \beta\, dx. \tag{1.15}$$

Let $\mathcal{H}_1$ denote the subset of $L^\infty(B)$ defined in (0.2). Then there exists a unique ρ, $0 \le \rho \le R$, so if $H_0 = \beta\chi_{B_\rho} + \alpha\chi_{B/B_\rho}$, where B_ρ denotes the ball centered at the origin of radius ρ, then $H_0 \in \mathcal{H}_1$ and

$$J_B(H) \le J_B(H_0) \quad \text{for all } H \in \mathcal{H}_1 \tag{1.16}$$

with equality if and only if $H = H_0$.

PROOF. Let $r = |x|$ denote the radial variable in Ω. We observe that there exists a unique $\rho \in [0, R]$ so the function $H_0(x) = \beta(x)\chi_{B_\rho}(x) + \alpha(x)\chi_{B/B_\rho}(x)$ is in $\mathcal{H}_1$. Indeed, the function

$$\phi(\rho) = \int_{B_\rho} \beta\, dx + \int_{B/B_\rho} \alpha\, dx, \qquad 0 \le \rho \le R,$$

is a strictly increasing absolutely continuous function, and since $\phi(0) \le \gamma \le \phi(R)$ by (1.15), there exists a unique ρ so $\phi(\rho) = \gamma$.

Now let $H \in \mathcal{H}_1$. Since α and β are radial and decreasing, we have $\alpha = \alpha^\#$ and $\beta = \beta^\#$ a.e. in B, and one verifies easily that since $\alpha \le H \le \beta$, we have $\alpha^\# \le H^\# \le \beta^\#$. Since $\int_B H\, dx = \int_B H^\#\, dx$, we have $H^\# \in H_1$, and by Theorem 1.1, $J(H) \le J(H^\#)$. This observation enables us to reduce our consideration to the case that $H = H(r)$ is a radial function which is decreasing for $0 \le r \le R$. If $H \in \mathcal{H}_1$ is such a function, we show that $J(H) \le J(H_0)$ with equality only if $H = H_0$. This will complete the proof of Theorem 1.2.

Thus assume that $H \in \mathcal{H}_1$ is radial and decreasing. Using (1.7) we calculate that

$$J(H_0) - J(H) = \frac{1}{nR^n} \int_0^R r^{n+1} \frac{(H - H_0)}{H_0 H}\, dr.$$

Since $\int_0^R H r^{n-1}\, dr = \int_0^R H_0 r^{n-1}\, dr$, we may rewrite this equation as

$$J(H_0) - J(H) = \frac{1}{nR^n} \int_0^R r^{n-1}(H - H_0)\left(\frac{r^2}{HH_0} - c\right) dr, \tag{1.17}$$

where c is any constant. Since r^2/HH_0 is increasing on $(0, R)$, we may choose c so that

$$\lim_{r \to \rho^-} \frac{r^2}{H(r)H_0(r)} \le c \le \lim_{r \to \rho^+} \frac{r^2}{H(r)H_0(r)}.$$

Then since $(H - H_0)(r) \leq 0$ if $r < \rho$ and $(H - H_0)(r) \geq 0$ if $r > \rho$, we find that the integrand in (1.17) is nonnegative, so $J(H_0) \geq J(H)$.

If $J(H) = J(H_0)$ for some $H \neq H_0$, then we see from (1.17) that $HH_0 = r^2/c$ on the set where $H \neq H_0$. Thus HH_0 is strictly increasing on a set of positive measure, which is a contradiction. Thus $J(H) = J(H_0)$ implies $H = H_0$ a.e., and the proof is complete.

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