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MARX-STROHHÄCKER DIFFERENTIAL SUBORDINATION SYSTEMS

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ABSTRACT. Let $f(z) = z + a_2z^2 + \cdots$ be analytic in the unit disc U and let $k(z) = z/(1-z)$. The classic Marx-Strohhäcker result, that a convex (univalent) function f is starlike of order $\frac{1}{2}$, can be written in terms of differential subordinations as

$$zf''(z)/f'(z) \prec zk''(z)/k'(z) \;\Rightarrow\; zf'(z)/f(z) \prec zk'(z)/k(z).$$

The authors determine general conditions on k for which this relation holds. They also determine a different set of general conditions on k for which

$$zf'(z)/f(z) \prec zk'(z)/k(z) \;\Rightarrow\; f(z)/z \prec k(z)/z.$$

Finally, differential subordinations with starlike superordinate functions are considered.

1. Introduction. Let f be analytic in the unit disc $U = \{z\colon |z| < 1\}$. The function f, with $f'(0) \neq 0$, is *convex* (*univalent*) if and only if $\operatorname{Re}[zf''(z)/f'(z)+1] > 0$ in U. The function f, with $f(0) = 0$ and $f'(0) \neq 0$, is *starlike* (*univalent*) if and only if $\operatorname{Re}[zf'(z)/f(z)] > 0$ in U. If, in addition, $\operatorname{Re}[zf'(z)/f(z)] > \frac{1}{2}$ then f is said to be starlike of order $\frac{1}{2}$ [**1**, Vol. 1, p. 137].

The classic result of Marx [**3**] and Strohhäcker [**10**] asserts that a convex function is starlike of order $\frac{1}{2}$, that is

$$\operatorname{Re}[zf''(z)/f'(z) + 1] > 0,\ z \in U \;\Rightarrow\; \operatorname{Re}[zf'(z)/f(z)] > \tfrac{1}{2},\ z \in U. \tag{1}$$

We can simplify this relation by expressing it in terms of subordination. If F and G are analytic in U, then F is *subordinate* to G, written $F \prec G$ or $F(z) \prec G(z)$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$.

If we let $k(z) = z/(1-z)$, then (1) can be rewritten as

$$zf''(z)/f'(z) \prec zk''(z)/k'(z) \;\Rightarrow\; zf'(z)/f(z) \prec zk'(z)/k(z). \tag{2}$$

This *Marx-Strohhäcker differential subordination system of first type* holds for other functions k; in [**5**, p. 194] and in [**7**, Corollary 2.3] it is shown that (2) also holds when $k(z) = z/(1-z)^2$ is the Koebe function. In §2 of this article we determine general conditions on k for which (2) holds.

If k is the Koebe function, then Marx [**3**] and Strohhäcker [**10**] also showed that

$$zf'(z)/f(z) \prec zk'(z)/k(z) \;\Rightarrow\; f(z)/z \prec k(z)/z. \tag{3}$$

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§3 deals with general first order differential subordinations of the form $\psi(p(z), zp'(z)) \prec h(z)$, where h is a starlike function. In addition, as a special application of these results we determine conditions on k for which the *Marx-Strohhäcker differential subodination system of second type*, as given in (3), holds.

We close this section with two lemmas that will be used in subsequent sections.

LEMMA 1. *Let F be analytic in U and let G be analytic and univalent on $\overline{U}$, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in U$ and $\varsigma_0 \in \partial U$, and an $m \geq 1$ for which $F(|z| < |z_0|) \subset G(U)$,*

(i) *$F(z_0) = G(\varsigma_0)$, and*

(ii) *$z_0 F'(z_0) = m\varsigma_0 G'(\varsigma_0)$.*

The proof of a more general form of this lemma may be found in [**4**, Lemma 1].

The second lemma deals with the concept of a subordination chain. A function $L(z,t)$, $z \in U$, $t \geq 0$, is a *subordination chain* if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $L(z, t_1) \prec L(z, t_2)$ when $0 \leq t_1 \leq t_2$.

LEMMA 2 [**8**, p. 159]. *The function $L(z,t) = a_1(t)z + \cdots$, with $a_1(t) \neq 0$ for all $t \geq 0$, is a subordination chain if and only if* $\mathrm{Re}[z(\partial L/\partial z)/(\partial L/\partial t)] > 0$ *for $z \in U$ and $t \geq 0$.*

2. Marx-Strohhäcker differential subordination systems of first type.

THEOREM 1. *Let q be univalent in U, with $q(0) = 1$. Set $Q(z) = zq'(z)/q(z)$, $h(z) = q(z) + Q(z)$ and suppose that*

(i) *Q is starlike in U ($\log q$ is convex in U), and*

(ii) $\mathrm{Re}[zh'(z)/Q(z)] = \mathrm{Re}[q(z) + zQ'(z)/Q(z)] > 0$, $z \in U$.

If B is an analytic function on U such that

$$B(z) \prec q(z) + zq'(z)/q(z) = h(z), \tag{4}$$

then the analytic solution p of the differential equation

$$zp'(z) + B(z)p(z) = 1 \qquad (p(0) = 1) \tag{5}$$

satisfies $p \prec 1/q$.

PROOF. Conditions (i) and (ii) imply that the function h is univalent (close-to-convex) [**1**, Vol. 2, p. 2], $h(0) = 1$, and hence (4) is well defined with $B(0) = 1$. If we let $b(z) = \int_0^z [B(t) - 1]t^{-1}\,dt$, then (5) has an analytic solution given by $p(z) = \int_0^z e^{b(t)}\,dt/ze^{b(z)}$. Condition (i) implies that $q(z) \neq 0$ in U. Hence the function $1/q$ is analytic and univalent in U.

Without loss of generality we can assume q is univalent and $q(z) \neq 0$ on $\overline{U}$. If not, then we can replace q, B, and p by $q_r(z) = q(rz)$, $B_r(z) = B(rz)$, and $p_r(z) = p(rz)$ respectively, where $0 < r < 1$. These new functions satisfy the conditions of the theorem on $\overline{U}$. We would then prove $p_r \prec 1/q_r$, and by letting $r \to 1^-$ we obtain $p \prec 1/q$.

The function

$$L(z,t) = h(z) + tQ(z) = q(z) + (1+t)zq'(z)/q(z) \tag{6}$$

is analytic in U for all $t \geq 0$, and it is continuously differentiable on $[0, \infty)$ for all $z \in U$. Since $Q'(0) = q'(0) \neq 0$ we have $\partial L/\partial z(0,t) = q'(0)(2+t) \neq 0$ for $t \geq 0$. From (ii) we obtain

$$\operatorname{Re}\left[z\,\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right] = \operatorname{Re}\left[q(z) + (1+t)\frac{zQ'(z)}{Q(z)}\right] \geq \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0,$$

for $z \in U$ and $t \geq 0$. Hence by Lemma 2, $L(z,t)$ is a subordination chain and so we have $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (6) we have $L(z,0) = h(z)$ and so we obtain

$$L(\varsigma, t) \notin h(U) \tag{7}$$

for $|\varsigma| = 1$ and $t \geq 0$.

Now assume that $p \not\prec 1/q$. From Lemma 1 there exist points $z_0 \in U$ and $\varsigma_0 \in \partial U$, and an $m \geq 1$ such that $p(z_0) = 1/q(\varsigma_0)$ and $z_0 p'(z_0) = -m\varsigma_0 q'(\varsigma_0)/q^2(\varsigma_0)$. Using these results together with (5) and (6) we obtain

$$B(z_0) = \frac{1}{p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} = q(\varsigma_0) + \frac{m\varsigma_0 q'(\varsigma_0)}{q(\varsigma_0)} = L(\varsigma_0, m-1).$$

By using (7) we deduce that $B(z_0) \notin h(U)$, which contradicts (4). Hence $p \prec 1/q$, completing the proof of the theorem.

We now turn our attention to considering the Marx-Strohhäcker differential subordination system of first type as given in (2). If we let $p(z) = zf'(z)/f(z)$ and $q(z) = zk'(z)/k(z)$ then (2) becomes

$$p(z) + zp'(z)/p(z) \prec q(z) + zq'(z)/q(z) \Rightarrow p(z) \prec q(z).$$

This differential subordination result was considered in [**5**, Theorem 3]. Before using this general result to prove (2), however, we have to first prove that $zf'(z)/f(z)$ is analytic in U, a condition which does not follow directly from the left side of (2).

THEOREM 2. *Let q satisfy the conditions of Theorem 1 and let*

$$k(z) = z \exp \int_0^z [q(t) - 1]t^{-1}\,dt. \tag{8}$$

If $f(z) = z + a_2 z^2 + \cdots$ is analytic in U, and

$$zf''(z)/f'(z) \prec zk''(z)/k'(z), \tag{9}$$

then $zf'(z)/f(z)$ is analytic in U and $zf'(z)/f(z) \prec zk'(z)/k(z)$.

PROOF. From (8) we obtain

$$q(z) = \frac{zk'(z)}{k(z)} \quad \text{and} \quad h(z) = q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{zk''(z)}{k'(z)}.$$

Thus $zk'(z)/k(z)$ is univalent, and by (ii) of Theorem 1 we see that $zk''(z)/k'(z)$ is also univalent. Since condition (9) implies $f'(z) \neq 0$, the function $B(z) = 1 + zf''(z)/f'(z)$ is analytic in U, and satisfies $B \prec h$. For this particular B equation (5) has the analytic solution $p(z) = f(z)/zf'(z)$. Since all the conditions of Theorem 1 are satisfied, we have $p \prec 1/q$. Since $1/q \neq 0$ this implies that $p \neq 0$, and so $1/p(z) = zf'(z)/f(z)$ is analytic in U. In addition, from $p \prec 1/q$ and $1/q \neq 0$ we obtain $1/p \prec q$, which yields the desired conclusion $zf'(z)/f(z) \prec zk'(z)/k(z)$.

EXAMPLES. (a) The function $q(z) = [1+(1-2\alpha)z]/(1-z)$, with $0 \le \alpha < 1$, satisfies (i) and (ii) of Theorem 1. From Theorem 2 we obtain $k(z) = k(\alpha, z) = z/(1-z)^{2(1-\alpha)}$ and

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(\alpha,z)}{k'(\alpha,z)} = \frac{2z(1-\alpha)(2+(1-2\alpha)z)}{(1-z)(1+(1-2\alpha)z)}$$
$$\Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{zk'(\alpha,z)}{k(\alpha,z)} = \frac{1+(1-2\alpha)z}{1-z}.$$

The cases $\alpha = \frac{1}{2}$ and $\alpha = 0$ correspond to the functions $z/(1-z)$ and $z/(1-z)^2$ respectively, which were mentioned in §1.

(b) The function $q(z) = z/(1-e^{-z})$, which is a convex (univalent) function [**6**, p. 70] satisfies $q(0) = 1$ and

$$Q(z) = zq'(z)/q(z) = 1 + z/(1-e^z) = 1 - q(-z).$$

Since q is convex we conclude that Q is also convex. By using (1) we obtain $\operatorname{Re} zQ'(z)/Q(z) > \frac{1}{2}$, which implies condition (i) of Theorem 1. Condition (ii) also holds since

$$\operatorname{Re}[q(z) + zQ'(z)/Q(z)] \ge 1/(e-1) + 1/2 > 0.$$

Hence by Theorem 2 we obtain $k(z) = e^z - 1$ and

$$zf''(z)/f'(z) \prec zk''(z)/k'(z) = z$$
$$\Rightarrow zf'(z)/f(z) \prec zk'(z)/k(z) = z/(1-e^{-z}).$$

(c) The function $q(z) = [(1+z)/(1-z)]^\beta$, with $0 < \beta \le 1$, satisfies (i) and (ii) of Theorem 1 and by Theorem 2 we have

$$\frac{zf''(z)}{f'(z)} \prec \left[\frac{1+z}{1-z}\right]^\beta + \frac{z^2+2\beta z-1}{1-z^2} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left[\frac{1+z}{1-z}\right]^\beta.$$

(d) Employing the function $q(z) = e^{\lambda z}$ in Theorem 1 we obtain $Q(z) = \lambda z$ and $q(z) + zQ'(z)/Q(z) = e^{\lambda z} + 1$. Condition (i) is satisfied and condition (ii) will be satisfied if $\operatorname{Re}[e^{\lambda z} + 1] > 0$. A simple calculation shows that this condition holds for $|\lambda| \le r_0 = 2.183\ldots$, where $r_0 = y_0/\operatorname{SIN} y_0$ and y_0 is the smallest root of $\operatorname{COS} y + \exp(y \operatorname{COT} y) = 0$. Hence by Theorem 2, for $|\lambda| \le r_0 = 2.183\ldots$, we obtain $k(z) = z \exp \int_0^z (e^{\lambda t} - 1)t^{-1}\,dt$ and

$$zf''(z)/f'(z) \prec zk''(z)/k'(z) = e^{\lambda z} + \lambda z - 1$$
$$\Rightarrow zf'(z)/f(z) \prec zk'(z)/k(z) = e^{\lambda z}.$$

(e) The function $q(z) = 1 + \lambda z$, $|\lambda| \le 1$, satisfies the conditions of Theorems 1 and 2 and we obtain $k(z) = ze^{\lambda z}$ and

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(z)}{k'(z)} = \frac{\lambda z(2+\lambda z)}{(1+\lambda z)} \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} = 1 + \lambda z.$$

In the special case $\lambda = 1$ this simplifies to

$$zf''(z)/f'(z) \prec z(2+z)/(1+z) \Rightarrow |zf'(z)/f(z) - 1| < 1.$$

This is an improvement of the result $|f''(z)/f'(z)| \le 3/2 \Rightarrow |zf'(z)/f(z) - 1| < 1$ proved in [**7**, Theorem 3] and the result $|zf''(z)/f'(z)+1| < 2 \Rightarrow |zf'(z)/f(z)-1| < 1$ proved in [**5**, Theorem 5].

3. Differential subordinations with starlike superordinate function.

THEOREM 3. *Let $H(z) = z + a_2z^2 + \cdots$ be starlike in U and let $\psi\colon C^2 \to C$ be a function of the form $\psi(r,s) = w(r)\cdot s^\alpha$ with $\alpha > 0$. In addition, suppose that*

$$\psi(q(z), zq'(z)) = w(q(z))\cdot(zq'(z))^\alpha = H(z) \tag{10}$$

has a univalent solution q. If p is analytic in U, with $p(0) = q(0)$, then

$$w(p(z))\cdot(zp'(z))^\alpha \prec H(z) \Rightarrow p(z) \prec q(z). \tag{11}$$

PROOF. If $p \not\prec q$ then by Lemma 1 there exist points $z_0 \in U$ and $\varsigma_0 \in \partial U$, and an $m \geq 1$ such that $p(z_0) = q(\varsigma_0)$ and $z_0p'(z_0) = m\varsigma_0q'(\varsigma_0)$. Using these results in (10) we obtain

$$w(p(z_0))\cdot(z_0p'(z_0))^\alpha = w(q(\varsigma_0))\cdot(m\varsigma_0q'(\varsigma_0))^\alpha = m^\alpha H(\varsigma_0).$$

Since $H(U)$ is starlike and $m^\alpha \geq 1$ we obtain $w(p(z_0))\cdot(z_0p'(z_0))^\alpha \notin H(U)$. This contradicts (10) and so we must have $p \prec q$.

Note that (10) requires $\psi(q(0), 0) = 0$. The special case $\psi(r,s) = r^{1-\alpha}s^\alpha$, with $0 < \alpha \leq 1$, satisfies Theorem 3, with $p(0) = q(0)$, and was considered in [**4**, p. 169]. For the purposes of this paper, Theorem 3 is most interesting when $\psi(r,s) = s/r$. Applying the theorem in this case we obtain

THEOREM 4. *Let $H(z) = z + a_2z^2 + \cdots$ be starlike in U and suppose that*

$$Q(z) = \exp\int_0^z H(t)t^{-1}\,dt \tag{12}$$

is univalent. If p is analytic in U, with $p(0) = 1$, then

$$zp'(z)/p(z) \prec zQ'(z)/Q(z) = H(z) \;\Rightarrow\; p(z) \prec Q(z). \tag{13}$$

COROLLARY 4.1. *If H is a convex (univalent) and $\operatorname{Re} H(z) > -1$, then Q as given in (12) is univalent and (13) holds.*

PROOF. From (1) and (12) we obtain

$$\operatorname{Re}[1 + zQ''(z)/Q'(z)] = \operatorname{Re}[H(z) + zH'(z)/H(z)] > -1 + \tfrac{1}{2} = -\tfrac{1}{2},$$

which shows that Q is univalent [**9**, Corollary 3].

If we set $H(z) = q(z) - 1$ and $p(z) = f(z)/z$ we obtain

COROLLARY 4.2. *Let q be convex with $\operatorname{Re} q(z) > 0$ and let*

$$k(z) = z\exp\int_0^z [q(t) - 1]t^{-1}\,dt.$$

If $f(z) = z + a_2z^2 + \cdots$ is analytic in U, then

$$zf'(z)/f(z) \prec zk'(z)/k(z) \Rightarrow f(z)/z \prec k(z)/z.$$

REMARKS. 1. This relation is precisely the one that the Koebe function satisfied in (3) of §1. The corollary lists sufficient conditions on k to ensure that the Marx-Strohhäcker differential subordination system of second type holds.

2. The functions q used in Examples (a), (b), (c), and (e) also satisfy the conditions of the corollary, whereas q in (d) does not for $|\lambda| > 1$. As a result, for Example (a) we have

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(\alpha,z)}{k'(\alpha,z)} \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{zk'(\alpha,z)}{k'(\alpha,z)} \Rightarrow \frac{f(z)}{z} \prec \frac{k(\alpha,z)}{z},$$

where $k(\alpha,z) = z/(1-z)^{2(1-\alpha)}$ and $0 \le \alpha < 1$.

For Example (b) we have

$$zf''(z)/f'(z) \prec z \Rightarrow zf'(z)/f(z) < z/(1-e^{-z}) \Rightarrow f(z)/z < (e^z-1)/z,$$

and for Example (c) we have the extension

$$\frac{zf'(z)}{f(z)} \prec \left[\frac{1+z}{1-z}\right]^{\beta} \Rightarrow \frac{f(z)}{z} \prec \exp\int_0^z \left(\left[\frac{1+t}{1-t}\right]^{\beta} - 1\right) t^{-1}\,dt$$

for $0 < \beta \le 1$. In the special case $\beta = \frac{1}{2}$ this simplifies to

$$\frac{zf'(z)}{f(z)} \prec \left[\frac{1+z}{1-z}\right]^{1/2} \Rightarrow \frac{f(z)}{z} \prec \frac{2\exp(\mathrm{SIN}^{-1} z)}{1+(1-z^2)^{1/2}} = Q(z).$$

From the last corollary we know that $\mathrm{Re}(1 + zQ''(z)/Q'(z)) > -\frac{1}{2}$ and that Q is univalent. To show that the function Q is also convex it is sufficient to show $\mathrm{Re}(1 + zQ''(z)/Q'(z)) \ge 0$ for $|z| = 1$. Since

$$1 + zQ''(z)/Q'(z) = [(1+z)/(1-z)]^{1/2} - 1 + \tfrac{1}{2}[(1-z)^{-1} + (1-z^2)^{-1/2}],$$

if we let $z = e^{it}$, $0 \le t \le \pi$, we obtain

$$(1+z)/(1-z) = i\,\mathrm{COT}(t/2), \qquad 1 - z^2 = -2ie^{it}\mathrm{SIN}\,t,$$

and

$$\mathrm{Re}[1 + zQ''(z)/Q'(z)] = (\tfrac{1}{2}\mathrm{COT}(t/2))^{1/2} - \tfrac{3}{4} + [\mathrm{COS}(t/2) + \mathrm{SIN}(t/2)]/4(\mathrm{SIN}\,t)^{1/2}.$$

If we let $x = (\mathrm{COT}(t/2))^{1/2} \ge 0$, we obtain

$$\mathrm{Re}[1 + zQ''(z)/Q'(z)] = (5x^2 - 3\sqrt{2}x + 1)/4\sqrt{2}x > 0.$$

We close this section by considering another differential subordination having a starlike superordinate function. In [**4**, p. 170] and in [**2**, p. 192] it is shown that if h is convex and p is analytic in U, then

$$p(z) + zp'(z) \prec h(z) \Rightarrow p(z) \prec z^{-1}\int_0^z h(t)\,dt.$$

We can replace the superordinate function h with a function of the form $[(1+z)/(1-z)]^{\alpha}$. For $\alpha > 1$ this function is not convex, but is starlike with respect to 1.

THEOREM 5. *Let $\beta_0 = 1.21872\ldots$ be the solution of $\beta\pi = 3\pi/2 - \mathrm{TAN}^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + 2\,\mathrm{TAN}^{-1}\beta/\pi$ for $0 < \beta \le \beta_0$. If p is analytic in U, with $p(0) = 1$, then*

$$p(z) + zp'(z) \prec \left[\frac{1+z}{1-z}\right]^{\alpha} \Rightarrow p(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta}.$$

PROOF. Note that $\beta\pi \leq 3\pi/2 - \mathrm{TAN}^{-1}\beta$ and $\beta \leq 3 - \alpha$ for $0 < \beta \leq \beta_0$. If we let $h(z) = [(1+z)/(1-z)]^\alpha$ and $q(z) = [(1+z)/(1-z)]^\beta$ then $h(U)$ and $q(U)$ are domains given by the sectors $|\arg h| < \alpha\pi/2$ and $|\arg g| < \beta\pi/2$ respectively. We need to show that $p \prec q$. If this is not true, then there exist points $z_0 \in U$ and $\varsigma_0 \in \partial U$ such that $p(z_0) = q(\varsigma_0)$ and $p(|z| < |z_0|) \subset q(U)$. We need to consider separately the case $p(z_0) \neq 0$ which corresponds to a point on one of the rays on the sector $q(U)$, and the case $p(z_0) = 0$ which corresponds to the corner of the sector.

Case (i). If $p(z_0) \neq 0$ then $\varsigma_0 \neq \pm 1$. If we let $ri = (1+\varsigma_0)/(1-\varsigma_0)$ and use Lemma 1 we obtain

$$\begin{aligned} p(z_0) + z_0 p'(z_0) &= q(\varsigma_0) + m\varsigma_0 q'(\varsigma_0) = (ri)^\beta - (ri)^{\beta-1} m\beta(1+r^2)/2 \\ &= (ri)^\beta[1 + im\beta(1+r^2)/2r], \end{aligned}$$

and

$$\arg(p(z_0) + z_0 p'(z_0)) = \beta\pi/2 + \mathrm{TAN}^{-1}(m\beta(r+1/r)/2).$$

Since TAN^{-1} is an increasing function and $m \geq 1$, by considering the cases $r > 0$ and $r < 0$ we obtain

$$\beta\pi/2 + \mathrm{TAN}^{-1}\beta \leq |\arg(p(z_0) + z_0 p'(z_0))| \leq \beta\pi/2 + \pi/2.$$

Using the fact that $\beta\pi \leq 3\pi/2 - \mathrm{TAN}^{-1}\beta$ for $0 < \beta \leq \beta_0$ and the definition of α we obtain

$$\alpha\pi/2 \leq |\arg(p(z_0) + z_0 p'(z_0))| \leq 2\pi - \alpha\pi/2.$$

This implies that $p(z_0) + z_0 p'(z_0)$ lies outside the sector $h(U)$, and since this contradicts the hypothesis of the theorem we must have $p \prec q$.

Case (ii). The case $p(z_0) = 0$ can occur only if $\beta \geq 1$, since $\beta < 1$ implies that the sector angle of $q(U)$ is less than π and $p(|z| = |z_0|)$ cannot pass through such a corner without itself having a corner. If $p'(z_0) \neq 0$, then the smallest possible value of $\arg(z_0 p'(z_0))$ is given by $(2\pi - \beta\pi/2) - \pi/2$, which occurs when $p(|z| = |z_0|)$ is tangent to the lower ray of the sector $q(U)$. The largest possible value of $\arg(z_0 p'(z_0))$ is given by $\beta\pi/2 + \pi/2$, which occurs when $p(|z| = |z_0|)$ is tangent to the upper ray of the sector $q(U)$. Using these limitations and the fact that $\beta \leq 3 - \alpha$ we obtain

$$\alpha\pi/2 \leq \arg(p(z_0) + zp'(z_0)) \leq 2\pi - \alpha\pi/2,$$

which gives the same contradiction as Case (i). The subcase $p'(z_0) = 0$ also yields a contradiction since $p(z_0) + z_0 p'(z_0) = 0 \notin h(U)$.

REMARKS. If we take $\beta = 1$ in Theorem 5 we obtain $\alpha(1) = \frac{3}{2}$ and

$$p(z) + zp'(z) \prec \left[\frac{1+z}{1-z}\right]^{3/2} \Rightarrow p(z) \prec \frac{1+z}{1-z}.$$

In this case the angle of the sector $h(U)$ is $270°$. The case with the largest angle corresponds to $\alpha_0 = \alpha(\beta_0) = \alpha(1.218\ldots) = 1.781\ldots$. In this case the angle of the sector $h(U)$ is $320.62\ldots°$, while the angle of sector $q(U)$ is $219.36\ldots°$.

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