

A NOTE ON THE CONVOLUTION OF PROBABILITY MEASURES

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ABSTRACT. Simple proofs of the continuity of the convolution of measures for certain topologies are presented. In addition, results on the compactness of probability measures and on infinite convolutions of probability measures are obtained.

1. Introduction. In this paper we study the behavior of uniform measures under convolution, this being one of the prime reasons for their invention. The methods used also yield both new results and simple proofs of existing results for the more familiar classes of tight and τ -smooth measures.

§2 is an exposition of some of the basic properties of uniform measures. In §3 we prove the continuity of convolution on the class of uniform measures and obtain some results on compactness. Among the byproducts are a simple proof of the continuity of convolution on the class of τ -smooth measures and a new result on the relative compactness of tight measures. In §4 convergence principles are obtained for uniform measures and τ -smooth measures.

2. Preliminaries. Let (X, \mathcal{U}) be a uniform space; the class of uniform measures on (X, \mathcal{U}) is defined as follows. Let $\Delta(X, \mathcal{U})$ denote the space of all bounded uniformly continuous real-valued functions on (X, \mathcal{U}) and let $\Delta(X, \mathcal{U})$ be equipped with the supremum norm. A subset of $\Delta(X, \mathcal{U})$ will be called a U.E.B. set if it is uniformly equicontinuous and bounded. Let $\Delta^*(X, \mathcal{U})$ be the dual of $\Delta(X, \mathcal{U})$ and M_s the space of measures with finite support on X . The class of uniform measures on (X, \mathcal{U}) , denoted by $M_u(X, \mathcal{U})$, is defined as the closure of M_s in $\Delta^*(X, \mathcal{U})$ for the topology of uniform convergence on the U.E.B. subsets of $\Delta(X, \mathcal{U})$.

The following are some of the properties of uniform measures. These results can be found in [1, 6, 7, and 10].

PROPOSITION 2.1. *If (X, \mathcal{U}) is a complete metric space, then $M_u(X, \mathcal{U})$ is identical with $M_t(X)$, the class of bounded tight measures on X .*

DEFINITION. Let \mathcal{D} be a family of pseudometrics on a set X . We shall say that \mathcal{D} is directed if $\rho_1, \rho_2 \in \mathcal{D}$ implies the existence of a $\rho_3 \in \mathcal{D}$ such that for all $x, y \in X$

$$\max\{\rho_1(x, y), \rho_2(x, y)\} \leq \rho_3(x, y).$$

Let ρ be a pseudometric on a set X ; we shall let X_ρ denote the metric quotient of the pseudometric space (X, ρ) and \hat{X}_ρ denote the completion of X_ρ . In addition we shall let π_ρ denote the quotient map of X into \hat{X}_ρ .

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PROPOSITION 2.2. *Let $(\mathcal{X}, \mathcal{U})$ be a uniform space and let \mathcal{D} be a directed family of pseudometrics which generate \mathcal{U} . Let $\mu \in \Delta^*(\mathcal{X}, \mathcal{U})$. Then μ is a uniform measure if and only if $\pi_\rho \mu \in M_t(\hat{\mathcal{X}}_\rho)$ for each $\rho \in \mathcal{D}$.*

REMARK. From Proposition 2.2 we can deduce that $M_\tau \subset M_u$, where M_τ is the class of τ -smooth linear functionals on $\Delta(\mathcal{X}, \mathcal{U})$. It should be noted that there are spaces such that M_τ is strictly contained in M_u (see [10]).

We shall now state some results on convergence and compactness in M_u .

PROPOSITION 2.3. *On M_u^+ the topology of uniform convergence on the U.E.B. subsets of $\Delta(\mathcal{X}, \mathcal{U})$ coincides with the topology of weak convergence.*

PROPOSITION 2.4. *Let A be a subset of M_u^+ . Then A is weakly relatively compact in M_u if and only if $\pi_\rho A$ is tight on $\hat{\mathcal{X}}_\rho$ for each $\rho \in \mathcal{D}$, where \mathcal{D} is a directed family of pseudometrics which generates the uniformity on \mathcal{X} .*

Let us proceed to the case where \mathcal{X} has the additional structure of a group. In particular let us assume that \mathcal{X} is a Hausdorff topological abelian group. Let \mathcal{U} be the associated uniformity. For $f \in \Delta(\mathcal{X}, \mathcal{U})$ and $x \in \mathcal{X}$, let $(\tau_x f)(y) = f(x + y)$. If $\nu \in \Delta^*(\mathcal{X}, \mathcal{U})$, then $\nu(\tau_x f)$ is a bounded uniformly continuous function on $(\mathcal{X}, \mathcal{U})$ and therefore if μ is another functional in $\Delta^*(\mathcal{X}, \mathcal{U})$ we can form the expression $\mu[\nu(\tau_x f)]$. This expression defines a continuous linear functional on $\Delta(\mathcal{X}, \mathcal{U})$ which is called the convolution of μ and ν (see [8, p. 262]); that is $(\mu * \nu)(f) = \mu[\nu(\tau_x f)]$. This convolution operation, however, need not be commutative when considered on the whole space Δ^* . This drawback, among other reasons, led LeCam in [9] to invent uniform measures, a large class of linear functionals on which the above convolution operation is commutative.

LEMMA 2.5. *Let \mathcal{X} be a topological abelian group and let ρ be a uniformly continuous pseudometric on \mathcal{X} which is compatible with the group operation. Let μ and ν be linear functionals in $\Delta^*(\mathcal{X}, \mathcal{U})$. Then $\pi_\rho(\mu * \nu) = (\pi_\rho \mu) * (\pi_\rho \nu)$.*

PROOF. Let $f \in \Delta(\hat{\mathcal{X}}_\rho, \rho)$. Then

$$[\pi_\rho(\mu * \nu)](f) = (\mu * \nu)(f \circ \pi_\rho) = \mu[\nu(\tau_x(f \circ \pi_\rho))] \quad \text{where } x \in \mathcal{X}.$$

But $\pi_\rho \nu(\tau_{\pi_\rho x} f) = \nu(\tau_x(f \circ \pi_\rho))$. Therefore

$$\begin{aligned} [\pi_\rho(\mu * \nu)](f) &= \mu[\pi_\rho \nu(\tau_{\pi_\rho x} f)] \\ &= (\pi_\rho \mu)[(\pi_\rho \nu)(\tau_y f)] \quad \text{where } y \in \hat{\mathcal{X}}_\rho \\ &= [(\pi_\rho \mu) * (\pi_\rho \nu)](f). \end{aligned}$$

LEMMA 2.6. *Let \mathcal{X} and ρ be as in Lemma 2.5. Let μ be a linear functional in $\Delta^*(\mathcal{X}, \mathcal{U})$. Then $\pi_\rho \bar{\mu} = \overline{\pi_\rho \mu}$, where $\bar{\mu}(f) = \mu(\bar{f})$ and $\bar{f}(x) = f(-x)$.*

PROOF.

$$(\pi_\rho \bar{\mu})(f) = \bar{\mu}(f \circ \pi_\rho) = \overline{\mu(f \circ \pi_\rho)}, \quad \text{where } f \in \Delta(\hat{\mathcal{X}}_\rho, \rho).$$

But

$$\overline{f \circ \pi_\rho} = \bar{f} \circ \pi_\rho, \quad (\pi_\rho \bar{\mu})(f) = \mu(\bar{f} \circ \pi_\rho) = (\pi_\rho \mu)(\bar{f}) = \overline{(\pi_\rho \mu)}(f).$$

3. The convolution operator. We shall now obtain some results on the behavior of uniform measures under convolution. In this section \mathcal{X} will be a Hausdorff topological abelian group and \mathcal{U} the associated uniformity.

LEMMA 3.1. *Let μ_1, μ_2, ν_1 , and ν_2 be uniform measures. If ρ is a uniformly continuous pseudometric on \mathcal{X} and if $S_\rho = \{f: |f(x) - f(y)| \leq \rho(x, y) \text{ for all } x, y \in \mathcal{X}, \|f\| \leq 1\}$, then*

$$\begin{aligned} \sup_{f \in S_\rho} |(\mu_1 * \nu_1)(f) - (\mu_2 * \nu_2)(f)| \\ \leq \sup_{f \in \|\nu_1\| S_\rho} |\mu_1(f) - \mu_2(f)| + \sup_{f \in \|\mu_2\| S_\rho} |\nu_1(f) - \nu_2(f)|. \end{aligned}$$

PROOF. Let $f \in S_\rho$. Then

$$\begin{aligned} |(\mu_1 * \nu_1)(f) - (\mu_2 * \nu_2)(f)| \\ \leq |(\mu_1 * \nu_1)(f) - (\mu_2 * \nu_1)(f)| + |(\mu_2 * \nu_1)(f) - (\mu_2 * \nu_2)(f)| \\ = |\mu_1[\nu_1(\tau_x f)] - \mu_2[\nu_1(\tau_x f)]| + |\nu_1[\mu_2(\tau_x f)] - \nu_2[\mu_2(\tau_x f)]|. \end{aligned}$$

However $\nu_1(\tau_x f)$ belongs to $\|\nu_1\| S_\rho$ and $\mu_2(\tau_x f)$ belongs to $\|\mu_2\| S_\rho$, so

$$|(\mu_1 * \nu_1)(f) - (\mu_2 * \nu_2)(f)| \leq \sup_{f \in \|\nu_1\| S_\rho} |\mu_1(f) - \mu_2(f)| + \sup_{f \in \|\mu_2\| S_\rho} |\nu_1(f) - \nu_2(f)|.$$

The lemma now follows.

From this lemma we can obtain the following result.

THEOREM 3.2. *Let $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be nets in M_u and let $\{\nu_\alpha\}$ belong to a ball in M_u . Suppose $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ converge to the uniform measures μ and ν , respectively, for the topology of uniform convergence on the U.E.B. subsets of $\Delta(\mathcal{X}, \mathcal{U})$. Then $\mu_\alpha * \nu_\alpha$ converges to $\mu * \nu$ for the topology of uniform convergence on the U.E.B. subsets of $\Delta(\mathcal{X}, \mathcal{U})$.*

REMARK. Note that the measures in the above theorem need not be positive.

Recall that on M_u^+ the topology of weak convergence coincides with the topology of uniform convergence on the U.E.B. subsets of $\Delta(\mathcal{X}, \mathcal{U})$. Thus we have proved the following result.

PROPOSITION 3.3. *Let $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be nets of positive uniform measures. If μ_α and ν_α converge weakly to the uniform measures μ and ν respectively, then $\mu_\alpha * \nu_\alpha$ converges weakly to $\mu * \nu$.*

An immediate consequence is the following result which can be found in [4 and 5].

COROLLARY. *Let $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be nets of positive τ -smooth measures on \mathcal{X} . If μ_α and ν_α converge weakly to the τ -smooth measures μ and ν respectively, then $\mu_\alpha * \nu_\alpha$ converges to $\mu * \nu$.*

Let us now look at compactness in M_u^+ .

DEFINITION. A linear functional μ in $\Delta^*(\mathcal{X}, \mathcal{U})$ is called a mean if $\mu \geq 0$ (that is, $\mu(f) \geq 0$ for $f \geq 0$) and $\mu(1) = 1$. If a linear functional is both a mean and a uniform measure it is called a uniform mean.

PROPOSITION 3.4. *Let $\{\lambda_\alpha\}$, $\{\mu_\alpha\}$, and $\{\nu_\alpha\}$ be nets of uniform means on \mathcal{X} such that $\lambda_\alpha = \mu_\alpha * \nu_\alpha$ for each α . If the nets $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ are weakly relatively compact, then so is the net $\{\nu_\alpha\}$.*

PROOF. Let \mathcal{D} be a directed family of pseudometrics such that each pseudometric in \mathcal{D} is compatible with the group operation and such that \mathcal{D} generates the

topology on \mathcal{X} . Let $\rho \in \mathcal{D}$. Then $\pi_\rho \lambda_\alpha$, $\pi_\rho \mu_\alpha$, and $\pi_\rho \nu_\alpha$ are tight probability measures on $\hat{\mathcal{X}}_\rho$ and $\pi_\rho \lambda_\alpha = (\pi_\rho \mu_\alpha) * (\pi_\rho \nu_\alpha)$. Furthermore, since $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ are weakly relatively compact in M_u^+ , $\{\pi_\rho \lambda_\alpha\}$ and $\{\pi_\rho \mu_\alpha\}$ are tight on $\hat{\mathcal{X}}_\rho$. But if $\{\pi_\rho \lambda_\alpha\}$ and $\{\pi_\rho \mu_\alpha\}$ are tight on $\hat{\mathcal{X}}_\rho$, then $\{\pi_\rho \nu_\alpha\}$ is also tight on $\hat{\mathcal{X}}_\rho$ (see [11, Theorem 2.1, p. 58]). Thus, for each $\rho \in \mathcal{D}$, $\{\pi_\rho \nu_\alpha\}$ is tight on $\hat{\mathcal{X}}_\rho$ and so $\{\nu_\alpha\}$ is weakly relatively compact in M_u^+ .

COROLLARY. *Let $\{\lambda_\alpha\}$, $\{\mu_\alpha\}$, and $\{\nu_\alpha\}$ be nets of tight probability measures on \mathcal{X} such that $\lambda_\alpha = \mu_\alpha * \nu_\alpha$ for each α . If the nets $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ are weakly relatively compact in $M_t(\mathcal{X})$, then $\{\nu_\alpha\}$ is also weakly relatively compact in $M_t(\mathcal{X})$.*

First we need the following lemma.

LEMMA 3.5. *Let λ and μ be tight probability measures on \mathcal{X} and let ν be a uniform mean on \mathcal{X} . If $\lambda = \mu * \nu$, then ν is also tight.*

PROOF. The proof is similar to Lemma 3.1 of [2]. Also see [5].

PROOF OF THE COROLLARY. From Proposition 3.4 it follows that $\{\nu_\alpha\}$ is weakly relatively compact in M_u . Let ν be a limit point of $\{\nu_\alpha\}$. Then $\nu \in M_u$ and there exist limit points λ and μ of $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ respectively, which satisfy $\lambda = \mu * \nu$ by Proposition 3.3. But λ and μ are tight, therefore, by Lemma 3.5, ν is also tight. Thus $\{\nu_\alpha\}$ is weakly relatively compact in M_t .

REMARK. It can be easily shown that if $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ are tight, then $\{\nu_\alpha\}$ is also tight. The above corollary takes this a step further and shows that this result is still true when tightness is replaced by weak relative compactness. It should be noted that for a general topological space tightness and relative weak compactness are not equivalent concepts for subsets of M_t .

4. Infinite convolutions of uniform measures. Let us turn to linear spaces. For these spaces we shall obtain results for uniform measures and τ -smooth measures which are along the lines of Kloss's convergence principle. This principle has been proved for tight measures on topological groups (see [3, 5, and 13]).

First we need the following result which can be found in [12].

THEOREM 4.1. *Let \mathcal{X} be a topological group and let μ and ν be tight probability measures on \mathcal{X} . If $\mu * \nu = \mu$, then the closed subgroup generated by the support of ν is necessarily compact.*

The following result is actually a refinement, which is possible for linear spaces, of Proposition 5 of [13] or Theorem 2 of [5].

LEMMA 4.2. *Let \mathcal{X} be a metrizable linear space which need not be locally convex. Let $\{\mu_n\}$ be a sequence of symmetric tight probability measures on \mathcal{X} and let $\nu_\eta = \mu_1 * \mu_2 * \mu_3 * \dots * \mu_n$. If ν and ν' are any two tight limit points of $\{\nu_n\}$, then $\nu = \nu'$.*

PROOF OF LEMMA 4.2. Let ν and ν' be limit points of $\{\nu_n\}$. Since \mathcal{X} is metrizable we can find two subsequences of $\{\nu_n\}$, $\{\nu_{n_m}\}$, and $\{\nu_{n_p}\}$, such that ν_{n_m} converges weakly to ν and ν_{n_p} converges weakly to ν' .

Now $\nu_{n_p} = \nu_{n_m} * \nu_{n_m n_p}$ when $n_m < n_p$ and where $\nu_{n_m n_p} = \mu_{n_m+1} * \mu_{n_m+2} * \dots * \mu_{n_p}$. $\{\nu_{n_p}\}$ and $\{\nu_{n_m}\}$ are weakly relatively compact in M_t and so, by the corollary to Proposition 3.4, $\{\nu_{n_m n_p}\}$ is also weakly relatively compact in M_t . Therefore,

$\nu' = \nu * \lambda$, where λ is a limit point of $\nu_{n_m n_p}$. Since each μ_n is symmetric, it follows that ν' , ν , and λ are also symmetric. Similarly $\nu = \nu' * \lambda'$ where λ' is symmetric. It follows that $\nu = \nu * (\lambda * \lambda')$ and $\nu' = \nu' * (\lambda' * \lambda)$. By Theorem 4.1 the support of $\lambda * \lambda'$, $S_{\lambda * \lambda'}$, is contained in a compact subspace of \mathcal{X} . However \mathcal{X} is a Hausdorff topological vector space and therefore $\{0\}$ is the only compact subspace of \mathcal{X} . Hence $S_{\lambda * \lambda'} = \{0\}$. But $S_{\lambda * \lambda'} = \text{closure of } S_\lambda + S_{\lambda'}$ and so $S_\lambda + S_{\lambda'} = \{0\}$. However, as noted above, λ and λ' are symmetric which implies that $S_\lambda = S_{\lambda'} = \{0\}$. Therefore $\nu = \nu'$.

THEOREM 4.3. *Let \mathcal{X} be a Hausdorff topological vector space which need not be locally convex. Let $\{\mu_n\}$ be a sequence of symmetric uniform means on \mathcal{X} and let $\nu_n = \mu_1 * \mu_2 * \cdots * \mu_n$. If ν and ν' , two uniform means, are limit points of the sequence $\{\nu_n\}$, then $\nu = \nu'$.*

Furthermore if $\{\nu_n\}$ is weakly relatively compact in M_u , then $\{\nu_n\}$ converges.

PROOF. We shall prove this theorem by mapping \mathcal{X} into the metric spaces $\hat{\mathcal{X}}_\rho$, $\rho \in \mathcal{D}$, where \mathcal{D} is as before. Let $\rho \in \mathcal{D}$. Then $\pi_\rho \mu_k$ and $\pi_\rho \nu_k$, $k = 1, 2, 3, \dots$, are symmetric tight probability measures on $\hat{\mathcal{X}}_\rho$. Also $\pi_\rho \nu_n = (\pi_\rho \mu_1) * (\pi_\rho \mu_2) * \cdots * (\pi_\rho \mu_n)$ by Lemma 2.5. Suppose that ν and ν' , two uniform means, are limit points of $\{\nu_n\}$. Then $\pi_\rho \nu$ and $\pi_\rho \nu'$ are tight limit points of $\{\pi_\rho \nu_n\}$. By Lemma 4.2 it follows that $\pi_\rho \nu = \pi_\rho \nu'$ and therefore $\nu = \nu'$.

Suppose $\{\nu_n\}$ is weakly relatively compact in M_u . Then the limit points of $\{\nu_n\}$ are identical and therefore $\{\nu_n\}$ converges.

COROLLARY. *Let \mathcal{X} be a Hausdorff topological vector space which need not be locally convex. Let $\{\mu_n\}$ be a sequence of symmetric τ -smooth probability measures on \mathcal{X} and let $\nu_n = \mu_1 * \mu_2 * \cdots * \mu_n$. If ν and ν' are two τ -smooth limit points of $\{\nu_n\}$, then $\nu = \nu'$.*

Furthermore if $\{\nu_n\}$ is weakly relatively compact in M_τ , then $\{\nu_n\}$ converges.

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