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AN ALTERNATING PROCEDURE FOR OPERATORS ON L_p SPACES

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ABSTRACT. Let L_p be the usual Banach spaces over a σ-finite measure space. If $1 < p < \infty$ and $q = p(p-1)^{-1}$, then $\psi_p \colon L_p \to L_q$ denotes the duality mapping defined by the requirements that $(f, \psi_p f) = \|f\|_p^p = \|f\|_p \|\psi_p f\|_q$, $f \in L_p$. If $T \colon L_p \to L_p$ is a bounded linear operator, then $M(T) \colon L_p \to L_p$ is the mapping defined by $M(T) = \psi_q T^* \psi_p T$, where $T^* \colon L_q \to L_q$ is the adjoint of T. It is proved that if T_n is a sequence of operators on L_p such that $\|T_n\| \leq 1$ for all n, then $M(T_n \cdots T_2 T_1) f$ converges in L_p for all $f \in L_p$.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure. Let $L_p = L_p(X, \mathcal{F}, \mu)$, $1 < p < \infty$, denote the usual Banach spaces of functions. We are going to consider the complex case only, as the proofs are the same in the real case. If T is a bounded and linear operator on L_p, then $M(T)$ denotes the operator on L_p defined by

$$M(T)f = (T^*(Tf)^*)^*, \qquad f \in L_p,$$

which is, in general, nonlinear. Here $T^* \colon L_q \to L_q$, $q = p(1-p)^{-1}$, is the dual operator of T and * denotes the duality map from L_p to L_q or from L_q to L_p. More explicit definitions will be given below. Let $T_n \colon L_p \to L_p$, $n \geq 1$, be an arbitrary sequence of (linear) contractions (i.e. $\|T_n\| \leq 1$, $n \geq 1$). We will prove that $M(T_n \cdots T_2 T_1) f$ converges in L_p for any $f \in L_p$. A particular case of this result where $p = 2$ and $T_n = P_2 P_1$ for all n, with two projections P_2 and P_1 in L_2, is due to von Neumann [**11**]. If $f \geq 0$ and $T_n = T$ for all n, where T is positive, then our result gives that $(T^{*n}(T^n f)^{p-1})^{1/(p-1)}$ converges in L_p. Another simplification occurs if $p = 2$ and $T_n = T$ for all n. In this case we obtain the convergence of $T^{*n} T^n f$ in L_2. This simple special case has already been noticed and used in [**2**].

In general $M(T_n \cdots T_1) f$ does not converge pointwise, even if $T_n = T$ for all n. This is essentially known (see [**3**, **10**, **1**, and **6**]). In fact, it has been observed by Burkholder [**3**] that there is a selfadjoint contraction T on L_2 and an $f \in L_2$ such that $n^{-1} \sum_{i=0}^{n-1} T^i f$ does not converge pointwise. Hence either $M(T^n) f = T^{2n} f$ or $M(T^n) T f = T^{2n+1} f$ does not converge pointwise. If the operators are positive, then, under additional hypotheses, deep pointwise convergence theorems are known: Rota's theorem [**8**], Stein's theorem [**9**]; see also [**10** and **7**], Burkholder and Chow [**4**], Doob [**5**]. In a paper under preparation we will return to the problem of pointwise convergence of $M(T_n \cdots T_1) f$ where the T_n's are positive contractions.

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2. Preliminaries. We identify the dual space of L_p with L_q in the usual way, $1 < p < \infty$, $q = p(p-1)^{-1}$. If $f \in L_p$ and $g \in L_q$, then we let $(f,g) = \int f\bar{g}\,d\mu$. The duality mapping $\psi_p\colon L_p \to L_q$ is defined by

$$(\psi_p f)(x) = \begin{cases} |f(x)|^p(\overline{f(x)})^{-1} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We also write f^* for $\psi_p f$, $f \in L_p$. It is easy to see that $f^* \in L_q$ and also that

$$(f, f^*) = \|f\|_p^p = \|f\|_p \cdot \|f^*\|_q.$$

In fact these two equations define f^* by the uniqueness part of the Hölder inequality. Note that $\psi_q\colon L_q \to L_p$ is the inverse of ψ_p. We also write $\psi_q g = g^*$, $g \in L_q$, when the distinction between ψ_p and ψ_q is clear from the context. If $T\colon L_p \to L_p$ is a bounded linear operator, then the dual operator $T^*\colon L_q \to L_q$ is defined by the requirement that $(Tf, g) = (f, T^*g)$ for all $f \in L_p, g \in L_q$. Finally we define $M(T)\colon L_p \to L_p$ by

$$M(T)f = \psi_q[T^*\psi_p(Tf)] = (T^*(Tf)^*)^*, \qquad f \in L_p,$$

and note that $M(T)(\alpha f) = \alpha M(T)f$ for any scalar (complex number) α.

We need the fact that L_p is a uniformly convex Banach space, $1 < p < \infty$. We state this result as follows [**6**].

2.1 *Uniform convexity of* L_p. For each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|f - g\|_p < \varepsilon$ whenever $\|f\|_p = \|g\|_p = 1$ and $\|f + g\|_p > 2 - \delta$.

2.2 LEMMA. *For each $\varepsilon > 0$ there is an $\eta > 0$ such that $\|f - g\|_p < \varepsilon$ whenever $\|f\|_p \le 1$, $\|g\|_p \le 1$, and $\|f + g\|_p > 2 - \eta$.*

PROOF. Given $\varepsilon > 0$ choose $\delta > 0$ as in (2.1) corresponding to $\varepsilon/8$ and let $\eta = \min(1/2, \varepsilon/8, \delta/8)$. Let $\|f\|_p \le 1$, $\|g\|_p \le 1$, and $\|f + g\|_p > 2 - \eta$. Then $\|f\|_p \ge \|f + g\|_p - \|g\|_p > 1 - \eta \ge 1/2$ and $\|g\|_p > 1 - \eta \ge 1/2$. Let $f_0 = f\|f\|_p^{-1}$, $g_0 = g\|g\|_p^{-1}$. Then

$$\|f_0 - f\|_p = \|f\|_p(\|f\|_p^{-1} - 1) \le (1-\eta)^{-1} - 1 \le \eta(1-\eta)^{-1} \le 2\eta$$

and $\|g_0 - g\|_p \le 2\eta$. Hence

$$\|f_0 + g_0\|_p \ge \|f + g\|_p - \|f_0 - f\|_p - \|g_0 - g\|_p \ge 2 - 5\eta > 2 - \delta.$$

Then (2.1) gives that $\|f_0 - g_0\|_p < \varepsilon/8$. Hence

$$\|f - g\|_p \le \|f_0 - g_0\|_p + \|f_0 - f\|_p + \|g_0 - g\|_p < \varepsilon/8 + 4\eta < \varepsilon.$$

2.3 LEMMA (UNIFORM CONTINUITY OF ψ_p). *Given $\varepsilon > 0$ there is a $\lambda > 0$ such that $\|\psi_p f - \psi_p g\|_q < \varepsilon$ whenever $\|f\|_p \le 1$, $\|g\|_p \le 1$, and $\|f - g\|_p < \lambda$.*

This result follows from Mazur [**7**], since the duality map ψ may be seen to be a composition of Mazur's maps F from L_1 to L_q and G from L_p to L_1, each of which is uniformly continuous in the unit ball.

2.4 LEMMA. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|f^* - T^*(Tf)^*\|_q < \varepsilon$ whenever $\|f\|_p \le 1$ and $T\colon L_p \to L_p$ is a (linear) contraction such that $\|f\|_p - \|Tf\|_p < \delta$.*

PROOF. Given $\varepsilon > 0$ find a $\delta_1 > 0$ such that $\|g - h\|_q < \varepsilon$ whenever $\|g\|_q \le 1$, $\|h\|_q \le 1$, and $\|g + h\|_q > 2 - \delta_1$. Let $\delta_2 = (\varepsilon/4)^{q-1}$. Find a $\delta_3 > 0$ such that

$(1 - r/\delta_2)^p > 1 - \delta_1$ whenever $0 \le r \le \delta_3$. Let $\delta = \min(\delta_2, \delta_3)$. Assume that $T\colon L_p \to L_p$ is a contraction and $f \in L_p$ is such that $\|f\|_p \le 1$ and $\|f\|_p - \|Tf\|_p < \delta$. Distinguish two cases:

(i) If $\|f\|_p \le \delta_2$, then $\|Tf\|_p \le \delta$ and

$$\|f^* - T^*(Tf)^*\|_q \le \|f^*\|_q + \|(Tf)^*\|_q \le \delta_2^{p-1} + \delta_2^{p-1} = \frac{\varepsilon}{2} < \varepsilon.$$

(ii) Assume $\delta_2 < \|f\|_p$ (≤ 1). Then

$$\begin{aligned}(f, f^* + T^*(Tf)^*) &= (f, f^*) + (Tf, (Tf)^*) = \|f\|_p^p + \|Tf\|_p^p \\ &\ge \|f\|_p^p \left(1 + \left(1 - \frac{\delta}{\|f\|_p}\right)^p\right) \\ &\ge \|f\|_p^p \left(1 + \left(1 - \frac{\delta}{\delta_2}\right)^p\right) > \|f\|_p^p(2 - \delta_1).\end{aligned}$$

This means that

$$\|f^* + T^*(Tf)^*\|_q > \|f\|_p^{p-1}(2 - \delta_1) = \|f^*\|_q(2 - \delta_1).$$

Hence

$$\left\| \frac{f^*}{\|f^*\|_q} + \frac{T^*(Tf)^*}{\|f^*\|_q} \right\|_q > 2 - \delta_1.$$

Then, by the choice of δ_1, we see that

$$\left\| \frac{f^*}{\|f^*\|_q} - \frac{T^*(Tf)^*}{\|f^*\|_q} \right\|_q < \varepsilon,$$

noticing that $\|T^*(Tf)^*\|_q \le \|(Tf)^*\|_q = \|Tf\|_p^{p-1} \le \|f\|_p^{p-1} = \|f^*\|_q$. Hence

$$\|f^* - T^*(Tf)^*\|_q < \varepsilon \|f^*\|_q \le \varepsilon.$$

3. THEOREM. *Let $T_n\colon L_p \to L_p$, $n \ge 1$, be a sequence of (linear) contractions and let $f \in L_p$. Then $M(T_n \cdots T_1)f$ converges in L_p as $n \to \infty$.*

PROOF. Since $M(S)\alpha f = \alpha M(S)f$ for any operator S on L_p and for any scalar α, we will assume that $\|f\|_p \le 1$, without loss of generality. Let $\alpha_n = \|T_n \cdots T_1 f\|_p$. Since α_n is a nonincreasing sequence of nonnegative numbers, given any $\delta > 0$ there is an n_0 such that $0 \le \alpha_{n_0} - \alpha_n < \delta$ for all $n \ge n_0$. Given $\varepsilon > 0$ first choose $\varepsilon_1 > 0$ such that $\|u^* - v^*\|_p < \varepsilon$ whenever $\|u\|_q \le 1$, $\|v\|_q \le 1$, and $\|u - v\|_q < \varepsilon_1$. Then choose $\delta > 0$ such that $\|g^* - S^*(Sg)^*\|_q < \varepsilon_1$ whenever $\|g\|_p \le 1$, $S\colon L_p \to L_p$ is a contraction, and $\|g\|_p - \|Sg\|_p < \delta$. Then find n_0 such that $\alpha_{n_0} - \alpha_n < \delta$ whenever $n \ge n_0$. This n_0 depends on $f \in L_p$ and on the sequence T_n. Let $g = T_{n_0} \cdots T_1 f$ and let $S_n = T_n \cdots T_{n_0+1}$, $n > n_0$. Hence $\|g\|_p - \|S_n g\|_p < \delta$ for all $n > n_0$ and also $\|g\|_p \le 1$. Hence $\|g^* - S_n^*(S_n g)^*\|_q < \varepsilon_1$, by the choice of δ. Since $R = T_{n_0} \cdots T_1$ is a contraction on L_p we then have that $\|R^* g^* - R^* S_n^*(S_n g)^*\|_q < \varepsilon_1$ or that $\|(R^* g^*)^* - (R^* S_n^*(S_n g)^*)^*\|_p < \varepsilon$. But this is just

$$\|M(T_{n_0} \cdots T_1)f - M(T_n \cdots T_1)f\|_p < \varepsilon$$

whenever $n \ge n_0$. Hence $M(T_n \cdots T_1)f$ converges in L_p.

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