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A STEINHAUS TYPE THEOREM

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ABSTRACT. The sequence space Λ_r, $r \geq 1$ being a fixed integer, is defined as

$$\Lambda_r = \{x = \{x_k\} \in l_\infty,\ x_k \in K,\ k = 0, 1, 2, \ldots, |x_{k+r} - x_k| \to 0,\ k \to \infty\},$$

where K is a complete, nontrivially valued field and l_∞ is the space of bounded sequences with entries in K. In this paper, it is proved that given a regular matrix $A = (a_{nk})$, $a_{nk} \in K = \mathbf{R}$ or $\mathbf{C}$, there exists a sequence in $\Lambda_r - \bigcup_{i=1}^{r-1} \Lambda_i$ which is not A-summable. This is an improvement of the well-known Steinhaus theorem. It is, however, shown that this result fails to hold when K is a complete, nontrivially valued, nonarchimedean field, whereas it is known that the Steinhaus theorem continues to hold.

In this paper, K denotes $\mathbf{R}$, the field of real numbers or $\mathbf{C}$, the field of complex numbers, or a complete, nontrivially valued, nonarchimedean field. In the relevant context, we explicitly mention which field is chosen.

If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots,$ is an infinite matrix and $x = \{x_k\}, x_k \in K$, $k = 0, 1, 2, \ldots$, by the A-transform of x, we mean the sequence $\{(Ax)_n\}$ where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \qquad n = 0, 1, 2, \ldots,$$

assuming that the series on the right converge. x is said to be summable by the matrix method A or A-summable to s if

$$(Ax)_n \to s, \quad \text{as } n \to \infty.$$

If X, Y are sequence spaces with elements whose entries are in K and if $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$, is an infinite matrix, A is said to transform X to Y, written as $A \in (X, Y)$, if whenever $x = \{x_k\} \in X$, $(Ax)_n$ is defined, $n = 0, 1, 2, \ldots$, and $\{(Ax)_n\} \in Y$.

l_∞ denotes the set of all bounded sequences with entries in K. For $x = \{x_k\} \in l_\infty$, the norm of x is defined by

$$\|x\| = \sup_{k \geq 0} |x_k|. \tag{1}$$

l_∞ is seen to be a Banach space if $K = \mathbf{R}$ or $\mathbf{C}$ and a nonarchimedean Banach space (see e.g. [**2**] for the definition) if K is a complete, nontrivially valued, nonarchimedean field. c denotes the set of all convergent sequences with entries in K. With respect to the norm defined by (1), c is a closed subspace of l_∞.

If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$, $A \in (c, c)$, and $\lim_{n\to\infty}(Ax)_n = \lim_{k\to\infty} x_k$, A is called a regular or Toeplitz matrix and we write $A \in (c, c; P)$. A

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is called a Schur matrix if $A \in (l_\infty, c)$. The well-known Steinhaus theorem can now be symbolically written as

$$(c, c; P) \cap (l_\infty, c) = \emptyset.$$

In this paper, the space $\Lambda_r \subset l_\infty$ (see [**3**]) is studied in the context of the Steinhaus theorem. It may be recalled that the space Λ_r, $r \geq 1$ being a fixed integer, is the set of all $\{x_k\} \in l_\infty$ such that $|x_{k+r} - x_k| \to 0$, as $k \to \infty$. Λ_r is a closed subspace of l_∞ with respect to the norm defined by (1).

For results such as the Steinhaus theorem mentioned above, a reference would be [**4**].

The following result, improving the Steinhaus theorem, is proved here.

THEOREM 1. $(c, c; P) \cap (\Lambda_r - \bigcap_{i=1}^{r-1} \Lambda_i, c) = \emptyset$ *when* $K = \mathbf{R}$ *or* $\mathbf{C}$.

PROOF. Let $A = (a_{nk})$ be a regular matrix. Then we can choose two sequences of positive integers $\{n(m)\}, \{k(m)\}$, such that if

$$m = 2p, \quad n(m) > n(m-1), \quad k(m) > k(m-1) + (2m-5)r,$$

then

$$\sum_{k=0}^{k(m-1)+(2m-5)r} |a_{n(m),k}| < \frac{1}{16},$$

$$\sum_{k=k(m)+1}^{\infty} |a_{n(m),k}| < \frac{1}{16};$$

and if

$$m = 2p+1, \quad n(m) > n(m-1), \quad k(m) > k(m-1) + (m-2)r,$$

then

$$\sum_{k=0}^{k(m-1)+(m-2)r} |a_{n(m),k}| < \frac{1}{16},$$

$$\sum_{k=k(m-1)+(m-2)r+1}^{k(m)} |a_{n(m),k}| > \frac{7}{8},$$

$$\sum_{k=k(m)+1}^{\infty} |a_{n(m),k}| < \frac{1}{16}.$$

Define the sequence $x = \{x_k\}$ as follows: if $k(2p-1) < k \le k(2p)$, then

$$x_k = \begin{cases} \dfrac{2p-2}{2p-1}, & k = k(2p-1)+1, \\ 1, & k(2p-1)+1 < k \le k(2p-1)+r, \\ \dfrac{2p-3}{2p-1}, & k = k(2p-1)+r+1, \\ 1, & k(2p-1)+r+1 < k \le k(2p-1)+2r, \\ \vdots & \\ 1, & k(2p-1)+(2p-4)r+1 < k \\ & \quad \le k(2p-1)+(2p-3)r, \\ \dfrac{1}{2p-1}, & k = k(2p-1)+(2p-3)r+1, \\ \dfrac{2p-2}{2p-1}, & k(2p-1)+(2p-3)r+1 < k \\ & \quad \le k(2p-1)+(2p-2)r, \\ 0, & k = k(2p-1)+(2p-2)r+1, \\ \dfrac{2p-3}{2p-1}, & k(2p-1)+(2p-2)r+1 < k \\ & \quad \le k(2p-1)+(2p-1)r, \\ 0, & k = k(2p-1)+(2p-1)r+1, \\ \vdots & \\ \dfrac{1}{2p-1}, & k(2p-1)+(4p-6)r+1 < k \\ & \quad \le k(2p-1)+(4p-5)r, \\ 0, & k(2p-1)+(4p-5)r < k \le k(2p), \end{cases}$$

and if $k(2p) < k \le k(2p+1)$, then

$$x_k = \begin{cases} \dfrac{1}{2p}, & k(2p) < k \le k(2p)+r, \\ \dfrac{2}{2p}, & k(2p)+r < k \le k(2p)+2r, \\ \vdots & \\ \dfrac{2p-1}{2p}, & k(2p)+(2p-2)r < k \le k(2p)+(2p-1)r, \\ 1, & k(2p)+(2p-1)r < k \le k(2p+1). \end{cases}$$

We note that, if $k(2p-1) < k \le k(2p)$,

$$|x_{k+r} - x_k| < \frac{1}{2p-1},$$

while, if $k(2p) < k \le k(2p+1)$,

$$|x_{k+r} - x_k| < \frac{1}{2p}.$$

Thus $|x_{k+r} - x_k| \to 0$, as $k \to \infty$, showing that $x = \{x_k\} \in \Lambda_r$. However,

$$|x_{k+1} - x_k| = \frac{2p-2}{2p-1} \quad \text{if } k = k(2p-1)+(2p-3)r,\ p = 1, 2, \ldots.$$

Hence $|x_{k+1} - x_k| \nrightarrow 0$, as $k \to \infty$, and consequently $x \notin \Lambda_1$. In a similar manner, we can prove that $x \notin \Lambda_i$, $i = 2, 3, \ldots, r-1$. Thus $x \in \Lambda_r - \bigcup_{i=1}^{r-1} \Lambda_i$. Further

$$\left.\begin{aligned} |(Ax)_{n(2p)}| &< \tfrac{1}{16} + \tfrac{1}{16} = \tfrac{1}{8}, \\ |(Ax)_{n(2p+1)}| &> \tfrac{7}{8} - \tfrac{1}{16} - \tfrac{1}{16} = \tfrac{3}{4} \end{aligned}\right\}, \qquad p = 1, 2, \ldots,$$

which shows that $\{(Ax)_n\} \notin c$. This completes the proof of the theorem.

REMARK. In the context of Theorem 1, one may enquire whether a matrix which sums all sequences of 0's and 1's in $\bigcup_{r=1}^{\infty} \Lambda_r$ sums all sequences in $\bigcup_{r=1}^{\infty} \Lambda_r$. The answer to this query is, however, in the negative. For, if $x = \{x_k\}$ is a sequence of 0's and 1's in $\bigcup_{r=1}^{\infty} \Lambda_r$, then $x_{k+r} = x_k$ for $k \geq K_0$ and for some integer $r \geq 1$, i.e., x is eventually periodic and hence summable $(C, 1)$ as is seen directly or using the idea of almost convergence (see [**4**, p. 12]). The sequence

$$\{0; 1, 0; \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{1}{2}, 0; \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{4}, \tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0; \ldots\}$$

is in Λ_1 but it is not summable $(C, 1)$.

In the context of Theorem 1, Theorem 2 below indicates a deviation from the classical case in the nonarchimedean case.

THEOREM 2. *There exists a regular matrix with entries in Q_p, the p-adic field for a prime p, which sums all sequences in Λ_r for a fixed positive integer r, i.e.* $(c, c; P) \cap (\Lambda_r, c) \neq \emptyset$.

PROOF. Consider the matrix $A = (a_{nk})$, $a_{nk} \in Q_p, n, k = 0, 1, 2, \ldots$, where

$$\left.\begin{aligned} a_{nk} &= \frac{1}{r}, \quad k = n, n+1, \ldots, n+r-1 \\ &= 0 \quad \text{otherwise} \end{aligned}\right\}, \qquad n = 0, 1, 2, \ldots.$$

$$\begin{aligned} (Ax)_{n+1} - (Ax)_n &= \frac{(x_{n+1} + x_{n+2} + \cdots + x_{n+r}) - (x_n + x_{n+1} + \cdots + x_{n+r-1})}{r} \\ &= \frac{x_{n+r} - x_n}{r} \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

if $x = \{x_k\} \in \Lambda_r$. Then A sums all sequences in Λ_r. It is clear that A is regular.

I thank the referee for putting the remark preceding Theorem 2 in the proper perspective.

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