PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

SIMILARITY OF PARTS TO THE WHOLE FOR CERTAIN MULTIPLICATION OPERATORS

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ABSTRACT. We show that the Bergman shift B, multiplication by z on the Bergman space A^2, is similar to its part $B|_N$ if and only if $N = \varphi A^2$, where φ is a finite product of interpolating Blaschke products. In addition, we show that B is not unitarily equivalent to any of its parts. For the analytic Toeplitz operator T_f on H^2, we obtain that T_f is similar to each of its parts if and only if T_f is unitarily equivalent to each of its parts if and only if f is a weak-star generator of H^∞.

1. Introduction. Let T be a bounded linear operator on a Hilbert space H. If M is a closed subspace of H such that $TM \subset M$, then M is said to be invariant for T. The restriction of T to one of its *nonzero* invariant subspaces is said to be a part of T. In this paper, we consider the problem of determining which parts of the Bergman shift are similar to the Bergman shift and the problem of determining when an analytic Toeplitz operator is similar to each of its parts.

The operators $T_1\colon H_1 \to H_1$ and $T_2\colon H_2 \to H_2$ are similar, written $T_1 \approx T_2$, if there is an invertible operator $S\colon H_1 \to H_2$ such that $T_2S = ST_1$. If S is unitary, then T_1 and T_2 are said to be unitarily equivalent, written $T_1 \cong T_2$.

Recall that the Bergman space A^2 is the Hilbert space of those functions f analytic on the open unit disk U which satisfy

$$\|f\|_{A^2}^2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^2 r\,dr\,d\theta < \infty.$$

In the following section, we show that the Bergman shift B, multiplication by z on A^2, is similar to its part $B|_N$ if and only if $N = \varphi A^2$, where φ is a finite product of interpolating Blaschke products. We also show that B is not unitarily equivalent to any of its parts. This contrasts sharply with the situation for the unweighted shift, multiplication by z, T_z, on the Hardy space of U, H^2. For any $f \in H^\infty$, let $T_f\colon H^2 \to H^2$ be the analytic Toeplitz operator with symbol f; hence, $(T_f h)(z) = f(z)h(z)$.

PROPOSITION. *$T_z\colon H^2 \to H^2$ is unitarily equivalent to each of its parts.*

PROOF. By Beurling's Theorem [**1**] each nonzero invariant subspace for T_z on H^2 has the form φH^2 for some inner function φ. The operator $T_\varphi\colon H^2 \to \varphi H^2$ is unitary (an onto isometry) and $T_z|_{\varphi H^2} T_\varphi = T_\varphi T_z$. Hence, the unweighted shift is unitarily equivalent to each of its parts.

It is natural to ask when an analytic Toeplitz operator shares with the unweighted shift the property that it is unitarily equivalent to each of its parts. In fact, Wang

Received by the editors January 27, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B35, 47B37.

and Stampfli [9] have raised the following question: If $f \in H^\infty$ and T_f is unitarily equivalent to each of its parts, is f a weak-star generator of H^∞; i.e., are the polynomials in f weak-star dense in H^∞?

In §3, we answer the question of Wang and Stampfli in the affirmative by proving the following theorem.

THEOREM 3.1. *The following are equivalent.*
(a) *f is a weak-star generator of H^∞.*
(b) *T_f has the same invariant subspaces as T_z.*
(c) *T_f is unitarily equivalent to each of its parts.*
(d) *T_f is similar to each of its parts.*

The equivalence of (a) and (b) was established by Sarason [7]. That (b) implies (c) is an easy consequence of Beurling's Theorem. What appears to be new is that (c) and (d) are equivalent and that either implies (a).

2. Parts of the Bergman shift similar to the Bergman shift. Suppose $T\colon H \to H$ is similar to its part $T|_N$ so that there is an invertible operator $S\colon H \to N$ such that $T|_N S = ST$. Note that S viewed as an operator on H commutes with T and is bounded below. Recall that an operator S on H is bounded below if there is a constant $\delta > 0$ such that $\|Sf\| \geq \delta\|f\|$ for all $f \in H$; hence, S is bounded below if and only if it is 1-1 with closed range.

PROPOSITION 2.1. *If $S\colon H \to H$ is bounded below and commutes with T, then $\operatorname{Ran} S$ is invariant for T and $T|_{\operatorname{Ran} S} \approx T$. If S is an isometry which commutes with T, then $T|_{\operatorname{Ran} S} \cong T$.*

PROOF. Since S is bounded below, $\operatorname{Ran} S = SH$ is closed; and since T commutes with S, $TSH = STH \subset SH$. Thus $\operatorname{Ran} S$ is invariant for T. Now, $S\colon H \to \operatorname{Ran} S$ is invertible and $T|_{\operatorname{Ran} S} S = ST$; hence, $T \approx T|_{\operatorname{Ran} S}$. If S is an isometry, the map $S\colon H \to \operatorname{Ran} S$ is unitary.

For any $f \in H^\infty$ define $M_f\colon A^2 \to A^2$ to be the operator of multiplication by f on A^2. Let $C = \{\varphi \in H^\infty \colon \varphi A^2 \text{ is a nonzeroclosed subspace of } A^2\} = \{\varphi \in H^\infty\colon M_\varphi \text{ is bounded below}\}$.

PROPOSITION 2.2. *The Bergman shift $M_z\colon A^2 \to A^2$ is similar to its part $M_z|_N\colon N \to N$ if and only if $N = \varphi A^2$ for some $\varphi \in C$.*

PROOF. If $\varphi \in C$, then M_φ is bounded below and M_z restricted to $\operatorname{Ran} M_\varphi = \varphi A^2$ is similar to M_z by Proposition 2.1. Conversely, if $M_z|_N$ is similar to M_z, there is an invertible operator $S\colon A^2 \to N$ such that $M_z|_N S = SM_z$. Since the operator S commutes with M_z, there is a $\varphi \in H^\infty$ such that $S = M_\varphi$ [8, Theorem 3]. Now,

$$N = \operatorname{Ran} S = \operatorname{Ran} M_\varphi = \varphi A^2$$

and $\varphi \in C$ since $N = \varphi A^2$ is closed and nonzero.

Thus the problem of finding those parts of $M_z\colon A^2 \to A^2$ which are similar to M_z is reduced to that of finding those $\varphi \in H^\infty$ for which M_φ is bounded below. The following result of D. Leucking [5, Corollary 1] characterizes those φ for which M_φ is bounded below on A^2.

THEOREM 2.3 (LEUCKING). *Let $h \in H^\infty$. The operator $M_h\colon A^2 \to A^2$ is bounded below if and only if there are positive numbers r and δ such that $G = \{z\colon |h(z)| > r\}$ satisfies $m(G \cap D) \ge \delta m(U \cap D)$ for all disks D with center on the circle $|z| = 1$. (Here, m represents planar Lebesgue measure.)*

We give another characterization of those $h \in H^\infty$ for which M_h is bounded below on A^2. The characterization is based on the following result of McDonald and Sundberg [**6**, Proposition 22] (cf. also Horowitz [**4**, Theorem 2]).

THEOREM 2.4 (MCDONALD-SUNDBERG). *Let φ be an inner function. M_φ : $A^2 \to A^2$ is bounded below if and only if φ is a finite product of interpolating Blaschke products (f.p.i.b.p.).*

Any function $h \in H^\infty$ has the factorization $h = PSF$, where P is a Blaschke product, S is a singular inner function, and F is an outer function in H^∞. A consequence of Beurling's Theorem is that $\{p(z)F\colon p \text{ is a polynomial}\}$ is dense in H^2 for any outer function $F \in H^2$. Since convergence in H^2 implies convergence in A^2 and since H^2 is contained in A^2 as a dense subset (the polynomials are dense in A^2), we must have that $\{p(z)F\colon p \text{ is a polynomial}\}$ is dense in A^2 for any outer function $F \in H^2$. In particular, the operator $M_F\colon A^2 \to A^2$ has dense range provided $F \in H^\infty$ is outer.

COROLLARY 2.5. *Let $h \in H^\infty$. The operator $M_h\colon A^2 \to A^2$ is bounded below if and only if $h = \varphi F$, where $F, 1/F \in H^\infty$ and where φ is a f.p.i.b.p. In this case* $\operatorname{Ran} M_h = \varphi A^2$.

PROOF. If $h = \varphi F$ where $F, 1/F \in H^\infty$ and φ is a f.p.i.b.p., then $\operatorname{Ran} M_h = \varphi F A^2 = \varphi A^2$ is closed by Theorem 2.4. Hence, M_h is bounded below.

Now suppose that $M_h\colon A^2 \to A^2$ is bounded below. Let $h = \varphi SF$, where φ is a Blaschke product, S is a singular inner function, and F is an outer function in H^∞. Since $M_h = M_{\varphi S} M_F = M_F M_{\varphi S}$, both M_F and $M_{\varphi S}$ must be bounded below. By Theorem 2.4, $S \equiv 1$ and φ is a f.p.i.b.p. Now, in addition to being bounded below, M_F has dense range since $F \in H^\infty$ is outer. It follows that M_F is invertible, and hence, $F, 1/F \in H^\infty$.

The following theorem is an immediate consequence of Proposition 2.2 and Corollary 2.5.

THEOREM 2.6. *The Bergman shift $B\colon A^2 \to A^2$ is similar to its part $B|_N$ if and only if $N = \varphi A^2$, where φ is a f.p.i.b.p.*

It is clear from the proof of Proposition 2.2 that the problem of determining which parts of B are unitarily equivalent to B is equivalent to that of determining which of the operators $M_h\colon A^2 \to A^2$ are isometries. However, none of these operators other than M_1 is an isometry.

PROPOSITION 2.7. *The Bergman shift is not unitarily equivalent to any of its (proper) parts.*

PROOF. Suppse $B \cong B|_N$. By the proof of Proposition 2.2, $N = \operatorname{Ran} M_h$ for some isometry $M_h\colon A^2 \to A^2$ ($h \in H^\infty$). Since $\|1\|_{A^2} = 1$ and M_h is an isometry, $\|h\|_{A^2} = 1$. Now, for any $z \in U$, $|h(z)| \le \|M_h\| = 1$ [**3**, Lemma 11]. By the maximum modulus theorem, $h(z) \equiv 1$ or $|h(z)| < 1$ for all $z \in U$. If $h \equiv 1$, then

$N = \operatorname{Ran} M_1 = A^2$ and $B|_N$ is the "whole" Bergman shift. If $|h(z)| < 1$ for all $z \in U$, then

$$\|h\|_{A^2}^2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |h(re^{i\theta})|^2 r\, dr\, d\theta < 1,$$

a contradiction.

REMARKS. 1. Proposition 2.2 stated for the Bergman shift has an analogue for any shift operator $M_z\colon H^2(\beta) \to H^2(\beta)$ with C replaced by $C_\beta = \{\varphi \in H^\infty(\beta)\colon M_\varphi\colon H^2(\beta) \to H^2(\beta)$ is bounded below$\}$ (cf. [**8**] for notation).

2. Theorems 2.3, 2.4, and Corollary 2.5 remain valid if A^2 is replaced by any member A^p $(p > 0)$ of the Bergman family of spaces.

3. The author was led to consider the question of what parts of B are similar to B as a first step in solving the problem of grouping parts of B into equivalence classes under similarity (or unitary equivalence). This problem was raised by H. Bercovici and was related to the author by C. Cowen. It is easy to see that $B|_{N_2} \approx B|_{N_1}$ if $N_2 = \varphi N_1$ for φ a f.p.i.b.p., but it seems likely that there are other parts of B similar to $B|_{N_1}$.

3. Uniform operators. Adopting terminology used by Wang and Stampfli in [**9**], we say that an operator T on a separable infinite dimensional Hilbert space H is uniform provided T is unitarily equivalent to $T|_M$ for every infinite dimensional subspace $M \subset H$ invariant for T. Hence, for example, the unweighted shift is uniform. In [**9**], Wang and Stampfli obtain a general representation theorem for uniform operators. Using the representation theorem, they show that a uniform operator which has an invariant subspace of finite codimension must be an analytic Toeplitz operator [**9**, Theorem 5]. If $f \in H^\infty$ is nonconstant, then it is easy to see that any nonzero subspace invariant for T_f must be infinite dimensional. Hence, Theorem 3.1 below and the result of Wang and Stampfli combine to show that a uniform operator which has an invariant subspace of finite codimension must be either a constant multiple of the identity or an analytic Toeplitz operator whose symbol is a weak-star generator of H^∞.

THEOREM 3.1. *The following are equivalent.*
(a) *f is a weak-star generator of H^∞.*
(b) *T_f has the same invariant subspaces as T_z.*
(c) *T_f is unitarily equivalent to each of its parts.*
(d) *T_f is similar to each of its parts.*

PROOF. (a) $\Leftrightarrow$ (b) Sarason [**7**].

(b) $\Rightarrow$ (c). Each nonzero invariant subspace for T_f has the form φH^2 for some inner function φ (Beurling's Theorem). Now, $T_\varphi\colon H^2 \to \varphi H^2$ is unitary and $T_f|_{\varphi H^2} T_\varphi = T_\varphi T_f$. Hence, T_f is unitarily equivalent to each of its parts.

(c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (b). Since any subspace invariant for T_z is easily seen to be invariant for T_f, what we must show is that any subspace $M \subset H^2$ invariant for T_f is also invariant for T_z.

Let $\langle f \rangle$ be the closed subspace of H^2 generated by $\{1, f, f^2, \ldots\}$. $\langle f \rangle$ is invariant for T_f and $T_f|_{\langle f \rangle}$ is cyclic with cyclic vector 1. Since $T_f|_{\langle f \rangle}$ is similar to T_f, T_f is cyclic with, say, cyclic vector g.

Let $M \subset H^2$ be an arbitrary invariant subspace for T_f. We need to show that $zM \subset M$. Since T_f is similar to $T_f|_M$, there is an invertible operator $S\colon H^2 \to M$ such that $T_f|_M S = ST_f$. Since g is cyclic for T_f, Sg is cyclic for $T_f|_M$. Claim $zSg \in M$.

Singe g is cyclic for T_f, there is a sequence $\{q_n\}$ of polynomials such that $\|q_n(T_f)g - zg\| \to 0$. Since convergence in H^2 implies pointwise convergence on the unit disk U, $q_n(f) \to z$ pointwise on U. Now,

$$\|q_n(T_f|_M)Sg - S(zg)\| = \|Sq_n(T_f)g - S(zg)\| \to 0$$

by the continuity of S. It follows that $q_n(f)Sg \to S(zg)$ pointwise on U, but $q_n(f) \to z$ pointwise. Hence, $zSg = S(zg) \in M$.

We now see that the dense subspace of M, $\{p(f)Sg\colon p \text{ is a polynomial}\}$, is mapped into M under T_z. It follows that $zM \subset M$.

REMARKS. 1. Results related to those of this section but concerning normal operators and their nonnormal parts may be found in [**2**, §3].

2. The operators $T_1\colon H_1 \to H_1$ and $T_2\colon H_2 \to H_2$ are quasisimilar if there are injective operators with dense range $X_{12}\colon H_2 \to H_1$ and $X_{12}\colon H_1 \to H_2$ such that $X_{12}T_2 = T_1X_{12}$ and $X_{21}T_1 = T_2X_{21}$. It is not difficult to see how to modify the proof of Theorem 3.1, (d) $\Rightarrow$ (b), to show that if T_f is quasisimilar to each of its parts, then T_f and T_z have the same invariant subspaces. Hence "T_f is quasisimilar to each of its parts" may be added to the list of equivalent conditions given in Theorem 3.1.

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