PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

A SINGULAR SPACE RELATED TO THE POINT-OPEN GAME

H. J. K. JUNNILA, T. NOGURA AND R. TELGÁRSKY

ABSTRACT. The authors construct a completely regular space X_0 such that Player I has a winning strategy in the point-open game $G(X_0)$, but X_0 has no σ-closure-preserving cover by **C**-scattered closed sets.

The purpose of this paper is to present the following.

EXAMPLE. A completely regular space X_0 such that Player I has a winning strategy in the point-open game $G(X_0)$, but X_0 has no σ-closure-preserving cover by **C**-scattered closed sets.

Given a space X, the point-open game $G(X)$ is defined as follows: at move n, Player I chooses a point x_n in X, and then Player II chooses an open neighborhood U_n of x_n. Player I wins the play $(x_0, U_0, x_1, U_1, \dots)$ of $G(X)$ if $\bigcup\{U_n : n < \omega\} = X$ (see [**G**, **T**$_2$]). If instead of points x_n Player I chooses compact sets C_n, and if Player I has a winning strategy in the resulting game, then X is called compact-like (see [**T**$_2$, **T**$_3$]). If a point x in a space X has an open neighborhood U such that $\overline{U}$ is compact, then x is called a point of local compactness of X. A space X such that each nonempty closed subspace has a point of local compactness is called **C**-scattered [**T**$_1$]. A family $\mathcal{F}$ of subsets of X with the property that for any subfamily of $\mathcal{F}$, the closure and the sum commute, is called closure-preserving [**M**]. A countable union of closure-preserving families is called σ-closure-preserving.

The space X_0 in the above example is a modification of the space constructed by Nogura [**N**]; however, it improves his result in several aspects. Nogura showed, under the continuum hypotheses, that a compact-like space need not have a countable cover by **C**-scattered closed subsets, answering a question of Telgársky [**T**$_2$, **T**$_3$]. Here, (1) the continuum hypothesis is eliminated, (2) the game condition involving Player I is substantially strengthened, and (3) the countable union is generalized to a σ-closure-preserving union.

Since each compact space is **C**-scattered, it follows that X_0 has no σ-closure-preserving cover by compact sets. In contrast, a hereditarily metacompact compact-like space does have a closure-preserving cover by compact sets (see [**JST**]). Moreover, each compact-like space is the union of countably many **C**-scattered subsets (see [**T**$_3$]), where the **C**-scattered subsets cannot be closed in general (it follows from [**N**] and also from the above example).

Received by the editors December 18, 1985 and, in revised form, January 30, 1986. The contents of this paper have been presented to the American Mathematical Society at its 92nd Annual Meeting in New Orleans, Louisiana, January 7–11, 1986 (Session on General Topology, #569) by R. Telgársky.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54D20, 90D42, 54D30, 54G20.

Key words and phrases. Point-open game, compact-like space, closure-preserving, σ-closure-preserving, **C**-scattered, metacompact.

CONSTRUCTION. Let $T = \{t_\alpha : \alpha < \omega_1\}$ be a subset of the closed unit interval $[0,1]$, where $t_\alpha \neq t_\beta$ for $\alpha \neq \beta$. Let $T_0 = T \times \{\omega_1\}$ and

$$X = \{(t_\alpha, \beta) : \alpha < \beta < \omega_1\} \cup T_0.$$

The space X is a subspace of $[0,1] \times [0, \omega_1]$, where $[0, \omega_1]$ has the standard interval topology (that is, the topology generated by left-open right-closed intervals). The space X_0 is obtained from X by contracting the set T_0 into the singleton $\{x_0\}$.

CLAIM 1. Player I has a winning strategy in $G(X_0)$.

PROOF. Observe that the complement of any neighborhood of x_0 in X_0 is countable. Therefore, if the first move of Player I is x_0, he can easily take care of the remaining countable set.

For a subset H of X_0 and a $t \in T$ define

$$H_t = \{\alpha < \omega_1 : (t, \alpha) \in H\}.$$

CLAIM 2. If H is closed in X_0, then H_t is closed in $[0, \omega_1)$.

This is immediate, since the map which takes (t, α) to α is a homeomorphism between a subspace of X_0 and a closed subspace of $[0, \omega_1)$.

For a subset H of X_0 define

$$T_H = \{t \in T : H_t \text{ is uncountable}\}.$$

CLAIM 3. If H is closed in X_0, then T_H is closed in T.

PROOF. Let $t = \lim t(n)$, where $t \in T$ and $t(n) \in T_H$ for each $n < \omega$. Then each $H_{t(n)}$ is uncountable and closed in $[0, \omega_1)$, and thus $E = \bigcap\{H_{t(n)} : n < \omega\}$ is also uncountable and closed in $[0, \omega_1)$. Take an $\alpha < \omega_1$ such that $t_\alpha = t$. Then $E \cap (\alpha, \omega_1) \subset H_t$, and hence $t \in T_H$.

CLAIM 4. If $\mathcal{H}$ is a σ-closure-preserving family of closed subsets of X_0, then $\bigcup\{T_H : H \in \mathcal{H}\} = T_{\cup\mathcal{H}}$.

PROOF. Let $t \in T_{\cup\mathcal{H}}$. Then $(\bigcup \mathcal{H})_t$ is uncountable and is covered by the σ-closure-preserving closed family $\{H_t : H \in \mathcal{H}\}$. By Corollary 1 of Potoczny and Junnila [**PJ**], if each H_t were compact, then $(\bigcup \mathcal{H})_t$ would be the countable union of metacompact closed subsets of $[0, \omega_1)$. Since each metacompact closed subset of $[0, \omega_1)$ is countable, there is an $H \in \mathcal{H}$ such that H_t is not compact. Hence $t \in T_H$.

From Claims 3 and 4 we get

CLAIM 5. If $\mathcal{H}$ is a σ-closure-preserving closed cover of X_0, then $\{T_H : H \in \mathcal{H}\}$ is a σ-closure-preserving closed cover of T.

Let $\mathcal{H}$ be a σ-closure-preserving closed cover of X_0. We shall show in Claim 7 that some element of $\mathcal{H}$ is not **C**-scattered.

CLAIM 6. T_H is uncountable for some $H \in \mathcal{H}$.

PROOF. Since T is hereditarily separable, the σ-closure-preserving cover $\{T_H : H \in \mathcal{H}\}$ of T has a countable subcover. Hence the claim follows.

Let H be an element of $\mathcal{H}$ such that T_H is uncountable.

CLAIM 7. H is not **C**-scattered.

PROOF. Let $\{t(n) : n < \omega\}$ be a countable self-dense subset of T_H. Each $H_{t(n)}$ is a closed unbounded set in $[0, \omega_1)$, hence there exists an α in $\bigcap\{H_{t(n)} : n < \omega\}$. Let K be the closure of $\{(t(n), \alpha) : n < \omega\}$ in X_0. Then K is a countable self-dense closed subset of H. Hence H is not **C**-scattered.

REFERENCES

[**G**] F. Galvin, *Indeterminacy of point-open games*, Bull. Acad. Polon. Sci. **26** (1978), 445–449.

[**JST**] H. J. K. Junnila, J. C. Smith and R. Telgársky, *Closure-preserving covers by small sets*, Topology Appl. **23** (1986), 237–262.

[**M**] E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.

[**N**] T. Nogura, *A compact-like space which does not have a countable cover by C-scattered closed subsets*, Proc. Japan Acad. **59** (1983), 83–84.

[**PJ**] H. Potoczny and H. Junnila, *Closure-preserving families and metacompactness*, Proc. Amer. Math. Soc. **53** (1975), 523–529.

[$\mathbf{T}_1$] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. **73** (1971), 59–74.

[$\mathbf{T}_2$] ——, *Spaces defined by topological games*, Fund. Math. **88** (1975), 193–223.

[$\mathbf{T}_3$] ——, *Spaces defined by topological games*, II, Fund. Math. **116** (1983), 189–207.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, 00100 HELSINKI 10, FINLAND

DEPARTMENT OF MATHEMATICS, EHIME UNIVERSITY, MATSUYAMA 790, JAPAN

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701