PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

A NOTE ON THE BORSUK-ULAM THEOREM

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ABSTRACT. Let $\mathcal{F}$ denote the set of all maps from S^n to $\mathbf{R}^n$ topologized by the usual metric, and $\mathcal{B}$ the set of all nonempty closed subsets of S^n invariant with respect to the antipodal map. Let $\beta: \mathcal{F} \to \mathcal{B}$ assign to each $f \in \mathcal{F}$ the set of all x for which $f(x) = f(-x)$. The largest topology on $\mathcal{B}$ for which β is continuous is identified: it is the upper semifinite topology.

Let d denote the usual Pythagorean metric on $\mathbf{R}^n$ or S^n. Denote by $\mathcal{F}$ the set of all maps from S^n to $\mathbf{R}^n$ and by $\mathcal{B}$ the set of all nonempty closed subsets of S^n which are invariant with respect to the antipodal map of S^n. The Borsuk-Ulam theorem asserts that for each $f \in \mathcal{F}$ there is some point $x \in S^n$ for which $f(x) = f(-x)$; it is readily shown that the set of all such points is a member of $\mathcal{B}$. Thus there is a function $\beta: \mathcal{F} \to \mathcal{B}$ defined by

$$\beta(f) = \{x \in S^n : f(x) = f(-x)\}.$$

This note considers the continuity of the function β.

Topologize $\mathcal{F}$ using the metric derived from the usual metric on $\mathbf{R}^n$ in the usual way, and $\mathcal{B}$ by the upper semifinite topology. The upper semifinite topology, defined by Michael in [**2**] on the collection of all nonempty closed subsets of a topological space, has as basis $\{V^\# : V \text{ is an open subset of } S^n\}$, where $V^\# = \{C \in \mathcal{B} : C \subset V\}$. This topology is very weak; it is not even T_1.

THEOREM. *The function $\beta: \mathcal{F} \to \mathcal{B}$ is continuous. Moreover when $\mathcal{F}$ has the usual metric topology, the upper semifinite topology is the largest topology on $\mathcal{B}$ for which β is continuous.*

PROOF. Firstly it is shown that β is continuous. Suppose $f \in \mathcal{F}$ and V is an open subset of S^n with $\beta(f) \subset V$. Let

$$\varepsilon = \min\{d(f(x), f(-x)) : x \in S^n - V\}.$$

Then $\varepsilon > 0$ and if $g \in \mathcal{F}$ is within $\varepsilon/2$ of f, then $\beta(g) \subset V$. Thus $\beta^{-1}(V^\#)$ is a neighborhood of f in $\mathcal{F}$.

To show that the upper semifinite topology is the largest topology, let $\mathcal{U} \subset \mathcal{B}$ be such that $\beta^{-1}(\mathcal{U})$ is open. It will be shown that $\mathcal{U}$ is open in the upper semifinite topology, i.e.

$$\forall C \in \mathcal{U},\ \exists \text{ open } V \subset S^n \text{ with } C \subset V \text{ and } V^\# \subset \mathcal{U}.$$

Let $C \in \mathcal{U}$.

Received by the editors December 10, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H25; Secondary 54C35, 54C60.
Key words and phrases. Borsuk-Ulam theorem, graph topology, upper semifinite topology.

For each $x \in S^n$, let $\sigma_x\colon S^n \to \mathbf{R}^n$ be some stereographic projection from $-x$ of the closed hemisphere centered at x onto B^n and if y is in the complementary hemisphere let $\sigma_x(y) = -\sigma_x(-y)$. Then $\sigma_x(y) = \sigma_x(-y)$ if and only if $y = \pm x$.

For each $x \in C$ define $f_x\colon S^n \to \mathbf{R}^n$ by $f_x(y) = d(y, C)\sigma_x(y)$. Then $f_x \in \mathcal{F}$ and $\beta(f_x) = C$ so $f_x \in \beta^{-1}(\mathcal{U})$. Thus $\exists \varepsilon_x > 0$ such that if $g \in \mathcal{F}$ is within $2\varepsilon_x$ of f_x, then $g \in \beta^{-1}(\mathcal{U})$. Let

$$U_x = \{y \in S^n \colon d(y, C) < \varepsilon_x\}$$

and choose $\delta_x \in (0, 1/2)$ so that $B(x; 2\delta_x) \subset U_x$, where $B(x; r)$ denotes the open ball in S^n of radius r. Let $\{B(x_i; \delta_{x_i})\colon i = 1, \ldots, m\}$ be a finite subcover of the open cover $\{B(x; \delta_x)\colon x \in C\}$ of C.

Set

$$V = \bigcap_{i=1}^{m} U_{x_i} \cap \left[\bigcup_{i=1}^{m} B(x_i; \delta_{x_i})\right].$$

The set V is open and contains C. Suppose $D \in V^{\#}$. Since $D \subset V$ and $D \neq \emptyset$, there is an index i with $D \cap B(x_i; \delta_{x_i}) \neq \emptyset$, say $x \in D \cap B(x_i; \delta_{x_i})$. Note also that $D \subset U_{x_i}$. Define $\varphi\colon S^n \to \mathbf{R}$ by

$$\varphi(y) = \max\{d(y, C), \varepsilon_{x_i}\}$$

and $\psi\colon S^n \to \mathbf{R}^n$ by

$$\psi(y) = \begin{cases} \sigma_{x_i}(y) & \text{if } y \in S^n - B(\pm x_i; 2\delta_{x_i}), \\ 0 & \text{if } y = \pm x, \\ \dfrac{d(\sigma_{x_i}(y), \sigma_{x_i}(\pm x))}{d(\sigma_{x_i}(z), \sigma_{x_i}(\pm x))}\sigma_{x_i}(z) & \text{if } y \in B(\pm x_i; 2\delta_{x_i}) - \{\pm x\}. \end{cases}$$

In the last line, $z \in \partial B(\pm x_i; 2\delta_{x_i})$ is chosen so that $\sigma_{x_i}(\pm x)$, $\sigma_{x_i}(y)$, and $\sigma_{x_i}(z)$ are collinear and in that order. It is readily checked that ψ is continuous and that $\psi(y) = \psi(-y)$ if and only if $y = \pm x$.

Define $g\colon S^n \to \mathbf{R}^n$ by

$$g(y) = \frac{d(y, D)}{d(y, D) + d(y, S^n - U_{x_i})}\varphi(y)\psi(y).$$

Then $g \in \mathcal{F}$ and is within $2\varepsilon_{x_i}$ of f_{x_i} so $g \in \beta^{-1}(\mathcal{U})$ and hence $\beta(g) \in \mathcal{U}$. Since $\beta(g) = D$ it follows that $D \in \mathcal{U}$ so $V^{\#} \subset \mathcal{U}$ as required. This completes the proof.

The theorem above may be compared with the results contained in [**1**], where cohomological bounds are obtained for sets derived from the Borsuk-Ulam sets of a parametrized family of maps.

It follows from the theorem that β is surjective, i.e. every nonempty closed subset of S^n which is invariant with respect to the antipodal map is the Borsuk-Ulam set of some map $S^n \to \mathbf{R}^n$. This fact is comparable with Theorem 1 of [**3**].

References

1. Jan Jaworowski, *A continuous version of the Borsuk-Ulam theorem*, Proc. Amer. Math. Soc. **82** (1981), 112–114.
2. Ernest Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.
3. Herbert Robbins, *Some complements to Brouwer's fixed point theorem*, Israel J. Math. **5** (1967), 225–226.

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