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FREE ACTIONS ON PRODUCTS OF EVEN DIMENSIONAL SPHERES

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ABSTRACT. We show that if G is a finite group acting freely on $\prod_{j=1}^{k} S^{2n_j}$ and if the induced action on mod 2 homology is trivial, then $G \cong (\mathbf{Z}_2)^l$ for some $l \leq k$. We also show that if G acts freely on $(S^{2n})^k$ and G is cyclic of order 2^l, then $2^{l-1} \leq k$.

It is the purpose of this note to prove the following.

MAIN THEOREM. *Suppose X is a finite CW-complex, the homotopy type of $\prod_{j=1}^{k} S^{2n_j}$ and G is a finite group acting freely and cellularly on X.*

(i) *If the induced G-action on $H_*(X;\mathbf{Z}_2)$ is trivial, then $G \cong (\mathbf{Z}_2)^l$ for some $l \leq k$.*

(ii) *If $n_1 = n_2 = \cdots = n_k$ and G is cyclic of order 2^l, then $2^{l-1} \leq k$.* □

REMARKS. (1) Part (i) resolves a conjecture we posed in [**1**].

(2) Part (ii) is sharp in the sense that there is a free action of $\mathbf{Z}_{2^l}$ on $(S^{2n})^{2^{l-1}}$. In fact, we may define an action of $\mathbf{Z}_{2m}$ on $(S^n)^m$ as follows: Let T generate $\mathbf{Z}_{2m}$ and define $T(x_1, \ldots, x_m) = (x_2, \ldots, x_m, -x_1)$. This gives an action that is free if and only if m is a power of 2 [**2**].

THEOREM A. *Suppose X is a finite CW-complex with $H_{\mathrm{odd}}(X;\mathbf{Q}) = 0$. If a finite group G acts freely and cellularly on X in such a way that the induced action on $H_*(X;\mathbf{Z}_2)$ is trivial, then G contains no elements of order* **4**. □

THEOREM B. (i) *There are no elements of order* 4 *in the kernel of the reduction homomorphism $p\colon \mathrm{GL}(n,\mathbf{Z}) \to \mathrm{GL}(n,\mathbf{Z}_2)$.*

(ii) *There are no elements of order 2^l in* $\mathrm{GL}(2^{l-1}-1,\mathbf{Q})$. □

PROOF OF THEOREM A. Let T be an element of order 4 in G. The induced map $T_*\colon H_*(X;\mathbf{Z}) \to H_*(X;\mathbf{Z})$ is an element of the reduction homomorphism by hypothesis and $T_*^4 = $ identity. According to Theorem B(i), $T_*^2 = $ identity. It follows that T^2 will induce the identity map on $H_*(X;\mathbf{Q})$, in which case the Lefshetz number $L(T^2) = \sum_i \dim H_i(X;\mathbf{Q}) \neq 0$. This contradicts the assumption that T^2 is fixed point free. □

PROOF OF THE MAIN THEOREM. (i) The Euler characteristics of X and X/G related to the order of G by the formula $\chi(X/G)\cdot|G| = \chi(X)$. Under the hypothesis, $\chi(X) = 2^k$; consequently $|G|$ divides 2^k. A finite 2-group that contains no elements of order 4 must be isomorphic to $(\mathbf{Z}_2)^l$ for some l.

(ii) Let n be the common value $n_1 = \cdots = n_k$ and suppose $G \cong \mathbf{Z}_{2^l}$, $l \leq k$, with generator T. The induced map $T^*\colon H^{2n}(X;\mathbf{Q}) \to H^{2n}(X;\mathbf{Q})$ may be regarded as

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an element of $\mathrm{GL}(k, \mathbf{Q})$. Furthermore T^* must have order 2^l, otherwise $T^{2^{l-1}}$ would act trivially on $H^*(X; \mathbf{Q})$ causing the Lefshetz number $L(T^{2^{l-1}})$ to be nonzero. According to Theorem B(ii), $k \geq 2^{l-1}$. $\square$

PROOF OF THEOREM B. (i) Let T be an element in the kernel of the reduction homomorphism and satisfying $T^4 = I$. Then we obtain the identity

$$(T^2 - I)(T^2 + I) = 0.$$

We claim that $\det(T^2 + I) \neq 0$. To see this, first note that we may write $T = 2N + I$ and, consequently, $T^2 = 4M + I$. Thus

$$\det(T^2 + I) = \det(4M + 2I) = 2^n \det(2M + I).$$

But $\det(2M + I)$ is necessarily an odd integer (expand by minors across the top row). Thus $T^2 + I$ is nonsingular and we obtain $T^2 = I$.

(ii) Suppose $T \in \mathrm{GL}(n, \mathbf{Q})$ has order 2^l. Diagonalize T over $\mathbf{C}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Each λ_j is a 2^lth root of unity. In fact, at least one λ_j must be primitive (else $T^{2^{l-1}} =$ identity). Thus, the characteristic polynomial for T must be divisible by the 2^lth cyclotomic polynomial of degree $\phi(2^l) = 2^{l-1}$. Consequently, $n \geq 2^{l-1}$. $\square$

REFERENCES

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