PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

A NOTE ON A SELECTION THEOREM

E. MICHAEL

ABSTRACT. A recent selection theorem of O. N. Kolesnikov is generalized and its proof is simplified.

1. Introduction. The following result was recently obtained by O. N. Kolesnikov [**1**, Theorem 1].

THEOREM 1.1 [**1**]. *Suppose X is completely regular, Y complete metric and locally pathwise connected, $C \subset X$ countable, and $\phi(x) \subset Y$ dense in Y for all $x \in C$. Then there exists a continuous $f\colon X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in C$.*[1]

The purpose of this note is to give a proof of this theorem which is slightly simpler than the proof in [**1**], and which yields the following generalization.

THEOREM 1.2. *Let $X, Y, C \subset X$ and $\phi(x) \subset Y$ be as in Theorem 1.1. Let $g\colon X \to Y$ be continuous with $g(X) \subset E$ for some closed, locally contractible $E \subset Y$, and let $A \subset X \backslash C$ be closed in X. Then there exists a continuous $f\colon X \to Y$ such that $f|A = g|A$ and $f(x) \in \phi(x)$ for $x \in C$. Moreover, given $\varepsilon > 0$ one can choose f so that $d(f, g) < \varepsilon$.*

Theorem 1.1 follows immediately from Theorem 1.2, with $A = \varnothing$, by taking g to be a constant map.

As Example 4.1 will show, the assumption in Theorem 1.2 that $g(X) \subset E$ for some closed, locally contractible $E \subset Y$ cannot be omitted. It is, however, not hard to show that it can be replaced by the assumption that X is locally contractible at every $x \in C$.

2. A lemma.

LEMMA 2.1. *Let X be completely regular, Y metric and locally pathwise connected, $A \subset X$ closed, $x^* \in X \backslash A$, and $D \subset Y$. Suppose also that $E \subset Y$ is closed and locally contractible and that $g\colon X \to E$ is continuous with $g(x^*) \in \overline{D}$. Then for each $\varepsilon > 0$ there exists a closed, locally contractible $F \subset Y$ and a continuous $f\colon X \to F$ such that $f|A = g|A$, $d(f, g) < \varepsilon$ and $f(x^*) \in D$.*

PROOF. Let $W = \{y \in Y\colon d(y, g(x^*)) < \frac{1}{2}\varepsilon\}$. Let $V \subset W$ be a pathwise connected neighborhood of $g(x^*)$ in Y such that $V \cap E$ is contractible over $W \cap E$, and pick $y^* \in V \cap D$. If $y^* \in E$, let $F = E$; if $y^* \notin E$, let J be an arc[2] in V

Received by the editors December 26, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C65; Secondary 54D05.
Key words and phrases. Continuous selections, locally pathwise connected, locally contractible.

[1]This had previously been proved in [**2**] under stronger hypotheses on X and Y.

[2]Recall that every pathwise connected Hausdorff space is arcwise connected (see, for example, [**3**, Corollary 31.6]).

from y^* to some $p \in E$ such that $J \cap E = \{p\}$, and let $F = E \cup J$. In either case, F is locally contractible and $V \cap F$ is contractible over $W \cap F$. Pick a continuous $u\colon (V \cap F) \times \mathbf{I} \to W \cap F$ such that $u(y,0) = y$ and $u(y,1) = y^*$.

Let U be open in X with $x^* \in U \subset \overline{U} \subset g^{-1}(V) \backslash A$, and let $h\colon X \to \mathbf{I}$ be continuous such that $h(x^*) = 1$ and $h(X \backslash U) = 0$. Define $f\colon X \to F$ by $f(x) = g(x)$ if $x \notin U$ and $f(x) = u(g(x), h(x))$ if $x \in \overline{U}$. It is easy to check that this f is well defined and satisfies our requirements. □

3. Proof of Theorem 1.2. We assume that C is infinite (the finite case is even simpler), and write $C = (x_n)$ with $x_m \neq x_n$ for $m \neq n$. Let $A_0 = A$ and $A_n = A \cup \{x_1, \dots, x_n\}$ for $n \geq 1$. Apply Lemma 2.1 inductively to construct closed, locally contractible $E_n \subset Y$ and continuous $f_n\colon X \to E_n$ such that $E_0 = E, f_0 = g$, and, for all $n \geq 0$, $f_{n+1}|A_n = f_n|A_n$, $d(f_{n+1}, f_n) < 2^{-n-1}\varepsilon$, and $f_{n+1}(x_{n+1}) \in \phi(x_{n+1})$. The sequence (f_n) is Cauchy, and thus converges uniformly to some $f\colon X \to Y$. This f has the required properties. □

REMARK. In both Lemma 2.1 and Theorem 1.2, the function f is actually constructed to be ε-homotopic to g.

4. An example. I am grateful to Joseph Martin for kindly suggesting the following example, communicated to me by Mary Ellen Rudin.

EXAMPLE 4.1. A Peano space[3] X, a closed $A \subset X$, an $x^* \in X \backslash A$, and a dense, open $D \subset X$ such that, if $f\colon X \to X$ is continuous and $f(x) = x$ for $x \in A$, then $f(x^*) \notin D$.

PROOF. Let Z be a Peano space which fails to be locally contractible at exactly one point p, and let $X = Z \times \mathbf{I}$. Let $A = Z \times \{0\}$, let $x^* = (p, 1)$, and let $D = (Z \backslash \{p\}) \times \mathbf{I}$. It is not hard to check that this works. □

REFERENCES

1. O. N. Kolesnikov, *Sections on set-valued mappings*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1984**, no. 2, 21–24 (= Moscow Univ. Math. Bull. **39** (1984), no. 2, 27–31).
2. E. Michael, *Continuous selections and countable sets*, Fund. Math. **111** (1981), 1–10.
3. S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

Current address (January–September 1987): Mathematisches Institut, Universität München, Theresienstrasse 39, D-8000 München, Federal Republic of Germany

[3] I.e., a connected, locally (pathwise) connected compact metric space.