PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 3, March 1987

ON THE EXISTENCE OF EXACTLY $(2,1)$ MAPS

R. E. SMITHSON

ABSTRACT. The following two theorems concerning the existence of exactly 2 to 1 maps are proved. If Y is a continuum such that each nondegenerate subcontinuum contains a cutpoint, then there does not exist a continuum X and an exactly 2 to 1 map on X onto Y. Further, if X is an arcwise connected continuum and Y is a nested continuum, then there does not exist an exactly 2 to 1 map on X onto Y.

1. Introduction. In 1940 O. G. Harrold [**5**] showed that the arc was not the image of an exactly n-to-1 map defined on a continuum. He also showed that an exactly n-to-1 image of a finite graph contains a circle. The natural question to ask here is which continua are the images of some continuum under an exactly n-to-1 map. So far the answers to this question are generally given in the negative as in the above-mentioned result of Harrold [**5**]. Recently Nadler and Ward [**6**] proved that if Y is a continuum such that every subcontinuum of Y contains an endpoint, then Y is not the image of a continuum under an exactly n-to-1 map. They were also able to show that if Y is a nonhereditarily unicoherent continuum, then there is a continuum X and an exactly n-to-1 map on X onto Y.

In the following, a *continuum* is a compact, connected T_2-space, and a *map* is a continuous function. Then a map f is exactly $(n,1)$ in case the inverse of each point in the range contains exactly n points. A *cutpoint* of a continuum Y is an element p such that $Y - \{p\}$ is not connected. Then an *arc* is a continuum which contains exactly two noncutpoints. In the sequel we shall be primarily interested in exactly $(2,1)$ maps.

2. Spaces with cutpoints. In this section we shall investigate the situation when the continuum Y is rich in cutpoints. First we present the following example.

EXAMPLE 1. Let X be the circle C_1, the twig at x_1 from C_1 to x_2 and C_2, C_3 two circles which are tangent at x_2 and do not meet C_1, and let Y be the two circles Y_1, Y_2 which are tangent at y_1 (see Figure 1).

Define f by: First, $f(x_1) = f(x_2) = y_1$. Then choose f so that it is 1-to-1, and continuous on C_1 onto $Y_1 - \{y_1\}$ and so that f is 1-to-1, and continuous on $L_1 - \{x_1, x_2\}$ onto Y_1. Finally, f maps $C_1 - x_2$ and $C_3 - x_2$ in a 1-to-1 continuous manner onto $Y_2 - y_1$. Then f is a continuous, exactly $(2,1)$ map on X onto Y.

Next we give a lemma which is crucial to the main result of this section.

LEMMA 2.1. *Let $f\colon X \to Y$ be an exactly $(2,1)$ map on the continuum X onto the continuum Y. If Y contains a cutpoint, then there are proper subcontinua X_0, Y_0 respectively such that f restricted to X_0 is exactly $(2,1)$ onto Y_0.*

Received by the editors March 20, 1984 and, in revised form, April 2, 1985.
1980 *Mathematics Subject Classification*. Primary 54C10, 54F20.
Key words and phrases. Exactly n-to-1 maps, continua, cutpoints, nested continua.

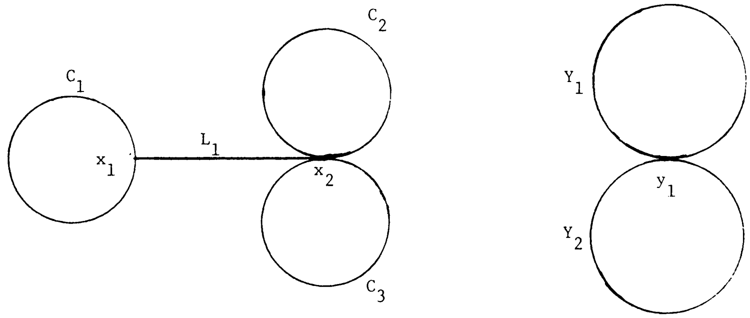

FIGURE 1

PROOF. Let p be a cutpoint of Y and let $Y - \{p\} = A \cup B$, where A, B are nonempty separated sets. Then $f^{-1}(p) = \{x_1, x_2\}$ and let M be a subcontinuum of X which is irreducible about x_1, x_2. Then $M - \{x_1, x_2\}$ is connected. Thus there is a component C_0 of $X - \{x_1, x_2\}$ which contains $M - \{x_1, x_2\}$. Now $f(C_0)$ is contained in either A or B. So assume that $f(C_0) \subset A$. Also note that $\overline{C}_0 = C_0 \cup \{x_1, x_2\}$. Next suppose that x is an element of $f^{-1}(A)$ and let C_x be the component of $X - \{x_1, x_2\}$ which contains x. Then $\overline{C}_x \cap \{x_1, x_2\} \neq \emptyset$ (see [**3**, Theorem 4.A.12, p. 105]) and $f(C_x) \subset A$.

Let $\mathbf{C}_1 = \{\overline{C}_x | x \in f^{-1}(A)$ and $x_1 \in \overline{C}_x\}$ and let $\mathbf{C}_2 = \{\overline{C}_x | x \in f^{-1}(A)$ and $x_2 \in \overline{C}_x\}$. Then $X_0 = \bigcup(\mathbf{C}_1 \cup \mathbf{C}_2) = f^{-1}(p \cup A)$ is closed and connected. Thus X_0 is a subcontinuum of X which contains C_0 and f restricted to X_0 maps onto $Y_0 = A \cup p$. Further, if $y \in A$ and $f^{-1}(y) = \{x, x'\}$, then $\{x, x'\} \in X_0$ and thus $f|_{X_0}$ is exactly $(2,1)$ on X_0. Moreover, Y_0 is a proper subcontinuum of Y since $B \neq \emptyset$, and we are done.

From Lemma 2.1 we get

THEOREM 2.2. *If Y is a continuum such that each nondegenerate subcontinuum of Y contains a cutpoint, then there does not exist a continuum X and an exactly $(2,1)$ map on X onto Y.*

PROOF. Suppose there is a continuum X and an exactly $(2,1)$ map on X onto Y. Let $\mathcal{A}$ be a maximal nest of subcontinua of X such that $f|_A$ is exactly $(2,1)$ for all $A \in \mathcal{A}$. Then $X_0 = \bigcap \mathcal{A}$ is a subcontinuum of X and $f|_{X_0}$ is exactly $(2,1)$. Now $Y_0 = f(X_0)$ is a nondegenerate subcontinuum of Y and, hence, contains a cutpoint. Then an application of Lemma 2.1 gives a contradiction to the maximality of $\mathcal{A}$.

The assumption that f was exactly $(2,1)$ was used strongly in the proof of Lemma 2.1 and the following example shows that the techniques used in that proof will not work for exactly $(n,1)$ maps when $n > 2$.

EXAMPLE 2. Let X be two thetas joined at a point as shown in Figure 2, and let Y be a "*figure* 8".

To define f, let $f(x_1) = f(x_2) = f(x_3) = p$ and let each of the open arcs with endpoints x_1, x_2 map homeomorphically onto $C_1 - \{p\}$. Also, let the open arcs with endpoints x_2, x_3 map homeomorphically onto $\mathbf{C}_2 - \{p\}$. Then f is an exactly $(3,1)$ map onto Y. However, if C is a component of $X - f^{-1}(p)$, then $\overline{C}$ does not contain

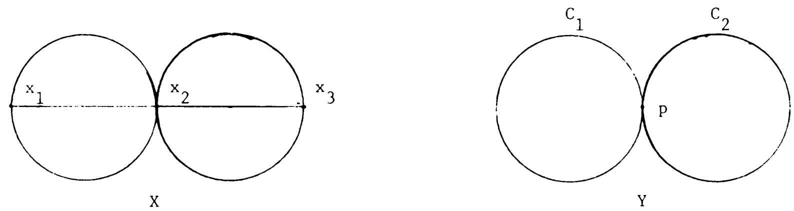

FIGURE 2

all of $f^{-1}(p)$. Thus the technique used in the proof of Lemma 2.1 will not yield a proper subcontinuum of X on which f is exactly $(3,1)$. There are subcontinua of X on which f is $(2,1)$ but so far a general method of finding such subcontinua has not been found. We are left with the following question.

Question 1. *Can Theorem* 2.2 *be extended to include exactly* $(n,1)$ *maps*?

An immediate corollary to Theorem 2.2 is Nadler and Ward's [**6**] theorem for $(2,1)$ maps. In Corollary 2.3 a point $e \in Y$ is an *endpoint* if e has arbitrarily small neighborhoods with one point boundary.

COROLLARY 2.3. *If Y is a continuum such that every subcontinuum of Y contains an endpoint, then there does not exist a continuum X and an exactly* $(2,1)$ *map on X onto Y.*

3. Arcwise connected shapes. If we put further restrictions on the domain space X, we can obtain other results. For example, it is known that there is no exactly $(2,1)$ map defined on a closed n-cell for $n = 1, 2$ or 3. In this section we consider the case when X is arcwise connected.

In the following, a space is *nested* in case it is an arcwise connected, Hausdorff space in which the union of a nest of arcs is contained in an arc.

THEOREM 3.1. *Let X be an arcwise connected continuum and Y a nested continuum. Then there does not exist an exactly* $(2,1)$ *map on X onto Y.*

PROOF. Suppose that $f\colon X \to Y$ is exactly $(2,1)$ and let $p \in Y$. Let $f^{-1}(p) = \{x_1, x_2\}$ and let A_1 be an arc in X with endpoints x_1, x_2. We know from Harrold's result [**4**] that f restricted to A_1 is not exactly $(2,1)$. Thus, let $x_3 \in A_1$ be such that $f^{-1}(f(x_3)) \cap A_1 = \{x_3\}$, and let $[p, f(x_3)]$ denote the unique arc in Y with endpoints $p, f(x_3)$. Then, by a result of Harris [**3**], $f(A_1)$ is arcwise connected and hence, $[p, f(x_3)] \subset f(A_1)$. Next we shall show that $f(A_1) \subset [p, f(x_3)]$. Suppose not; that is let $z \in A_1$ be such that $f(z) \notin [p, f(x_3)]$. Then there is an open interval (z_1, z_2) in A_1 such that $f((z_1, z_2)) \cap [p, f(x_3)] = \varnothing$ but $f(z_1)$ and $f(z_2)$ are in $[p, f(x_3)]$. We may assume that $[z_1, z_2] \subset [x_1, x_3]$ in A_1. Now suppose that $f(z_1) \neq f(z_2)$; then since $f(z_1), f(z_2) \in [p, f(x_3)]$, the unique arc $[f(z_1), f(z_2)] \subset [p, f(x_3)]$ and so $f(z_1, z_2)$ meets $[p, f(x_3)]$ which contradicts the choice of z_1, z_2. Thus $f(z_1) = f(z_2)$. The image of the arc $[x_2, x_3]$ must contain $[p, f(x_3)]$ and thus there is also a point x' in the arc $[x_2, x_3]$ in A_1 such that $f(x') = f(z_1) = f(z_2)$. This contradicts the fact that f is exactly $(2,1)$ on X. We have shown that each of the intervals $[x_1, x_3]$ and $[x_2, x_3]$ in A_1 maps onto $[p, f(x_3)]$ and f is exactly $(2,1)$ on $A - \{x_3\}$. Moreover, the above argument applies to any arc $A \subset X$ whose endpoints map onto the same point.

Now let $\mathcal{A}$ be a maximal nest of arcwise connected, closed subsets of X which satisfy the following conditions: (i) for each $A \in \mathcal{A}$, there is an $x_a \in A$ such that f is exactly $(2,1)$ on $A - \{x_a\}$; (ii) $f^{-1}(f(x_a)) \cap A = \{x_a\}$; and (iii) there is a $p \in Y$ such that $f(A) = [p, f(x_a)]$. By the above results we see that there are nonempty collections which satisfy the three conditions. Let $A_0 = \bigcup A$ and $X_0 = \overline{A}_0$. The collection $\{f(A) | A \in \mathcal{A}\}$ is a nest of arcs in Y and so there is a $y_m \in Y$ such that $f(A_0) \subset [p, y_m]$ and $\overline{f(A_0)} = [p, y_m]$. Further, $X_0 \subset f^{-1}([p, y_m])$.

Now let $f^{-1}(y_m) = \{s, z\}$ and let $x_1 \in f^{-1}(p) \cap A_0$. If A is an arc with endpoints x_1, s, then $f(A)$ contains $[p, y_m]$. Then, since $f^{-1}([p, y_m)) \subset A_0 \subset X_0$, there is a nest z_γ, $\gamma \in \Gamma$, in $A_0 \cap A$ such that $f(z_\gamma) \to y_m$. We may assume that $z_\gamma \to z$ in $A \cap X_0$, and, if $z \notin \{s, t\}$, then f is not exactly $(2,1)$. Hence, either s or t is in $A \cap X_0$. We assume that $s \in X_0$ and that $t \notin X_0$. We let A_1 be an arc with endpoints s, t. Then there is a point $a_1 \in A_1$ such that f is exactly $(2,1)$ on $A - \{a_1\}$ and $f^{-1}(f(a_1)) \cap A_1 = \{a_1\}$. Now f is exactly $(2,1)$ on A_0 and so f is exactly $(2,1)$ on $X_0 \cup A_1 - \{a_1\}$. If f is exactly $(2,1)$ on $X_0 \cup A_1$, then f is an exactly $(2,1)$ map on a continuum onto an arc, which is a contradiction. But $X_0 \cup A_1$ properly contains each member of $\mathcal{A}$ contrary to the maximality of $\mathcal{A}$. Finally, if both $x, t \in X_0$ we again have an exactly $(2,1)$ map on a continuum X_0 onto an arc. Thus there can be no such map on X.

We conclude with two questions.

Question 2. *Can the hypothesis that X and Y be continua in Theorem* 3.1 *be weakened? In particular, suppose that X is an arcwise connected T_2-space and Y a nested space. Then is there an exactly* $(2,1)$ *map on X onto Y?*

Question 3. *Is there an exactly* $(n,1)$ *version of Theorem* 3.1?

References

1. C. O. Christenson and W. L. Voxman, *Aspects of topology*, Marcel Dekker, New York, 1977.
2. P. Civin, *Two-to-one mappings of manifolds*, Duke Math. J. **10** (1943), 49–57.
3. J. K. Harris, *Order structures for certain acyclic topological spaces*, Thesis, Univ. of Oregon, 1962.
4. O. G. Harrold, *The non-existence of a certain type of continuous transformation*, Duke Math. J. **5** (1939), 789–793.
5. ——, *Exactly* $(k,1)$ *transformations on connected linear graphs*, Amer. J. Math. **62** (1940), 823-834.
6. S. B. Nadler, Jr. and L. E. Ward, Jr., *Concerning exactly* $(n,1)$ *images of continua*, Proc. Amer. Math. Soc. **87** (1983), 351–354.
7. J. H. Roberts, *Two-to-one transformations*, Duke Math. J. **6** (1940), 256–262.

Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071