THE SUP = MAX PROBLEM FOR $\delta$

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Abstract. Let $\delta(X) = \sup\{ d(D) : D \text{ is a dense subspace of } X \}$. It is shown that if $\kappa$ is a limit cardinal, but not a strong limit, and $\text{cf}(\kappa) > \omega$, then there is a 0-dimensional Hausdorff space $X$ such that $\delta(X) = \kappa$, but for all dense $D \subset X$, $d(D) < \kappa$. For all other values of $\kappa$, if $X$ is Hausdorff and $\delta(X) = \kappa$, then there is a dense $D \subset X$ such that $d(D) = \kappa$.

1. Introduction. We consider the SUP = MAX problem for the cardinal function $\delta$ defined as

$$\delta(X) = \sup \{ d(D) : D \text{ is a dense subspace of } X \}.$$ 

For Hausdorff spaces, the solution is given by Theorem 1.

It is easy to verify that $d(X) \leq \delta(X) \leq d(X) \cdot t(X)$. Let $X = 2^{\omega_1}$. Then $d(X) = \omega$. However, $\Sigma(2^{\omega_1}) \subset X$ is dense, and $d(\Sigma(2^{\omega_1})) = \omega_1 = \omega(X)$. Thus we have an example where $\delta(X) > d(X)$.

**Theorem 1.** If $\kappa$ is a limit cardinal, but not a strong limit, and $\text{cf}(\kappa) > \omega$, then there is a 0-dimensional Hausdorff space $X$ such that $\delta(X) = \kappa$, but for all dense $D \subset X$, $d(D) < \kappa$. Otherwise, if $X$ is Hausdorff and $\delta(X) = \kappa$, then there is a dense $D \subset X$ such that $d(D) = \kappa$.

We will prove Theorem 1 in §§2 and 3.

As always with the SUP = MAX problem, we need only consider the case where $\delta(X) = \kappa$ is a limit. It is easy to see that the theorem fails for non-Hausdorff $X$. Suppose, for example, $\kappa = \bigcup_{\alpha < \text{cf}(\kappa)} \kappa_\alpha$. Let $\{ X_\alpha : \alpha < \text{cf}(\kappa) \}$ be a pairwise disjoint collection of sets with $|X_\alpha| = \kappa_\alpha$. Let $X = \bigcup_{\alpha < \text{cf}(\kappa)} X_\alpha$. Define a set $O \subset X$ to be open if either $O = \emptyset$ or $|X_\alpha - O| < \kappa_\alpha$ for all $\alpha < \text{cf}(\kappa)$. $X$ is $T_1$ but not $T_2$. Since $X_\alpha$ is dense in $X$, $\delta(X) = \kappa$. If $D \subset X$ is dense, then $|D \cap X_\alpha| = \kappa_\alpha$ for some $\alpha < \text{cf}(\kappa)$ (otherwise $D$ is closed), and then $D \cap X_\alpha$ is dense, so $d(D) < \kappa$. Thus SUP = MAX fails for all limits.

We will use the following notation. If $S$ is a set, $\sigma(S) = \{ p \in 2^S : |p^-(1)| < \omega \}$. Note that $\sigma(S)$ is dense in $2^S$. If $S$ is a set, $H(S)$ is the collection of all finite partial functions from $S$ into $\{0,1\}$. If $h \in H(S)$, then $\langle h \rangle = \{ p \in 2^S : p \text{ extends } h \}$. Thus $\{ \langle h \rangle : h \in H(S) \}$ is the standard basis for $2^S$.

For the rest of the paper, we will assume that all spaces are Hausdorff.
2. When SUP = MAX. We first prove the second part of the theorem. As noted, we may assume \( \kappa \) is a limit cardinal. Suppose \( \delta(X) = \kappa \) and \( \kappa \) is a strong limit (i.e. if \( \lambda < \kappa \) then \( 2^\lambda < \kappa \)). Then \( d(X) = \kappa \) since \( |X| \leq \exp(\exp(d(X))) \) [1, Theorem 2.4] and \( \delta(X) \leq |X| \).

Suppose \( \delta(X) = \kappa \) and \( \text{cf}(\kappa) = \omega \). Let

\[ \mathcal{B} = \{ O \subset X: O \neq \emptyset \text{ is open and if } U \subset O \text{ is open then } \delta(U) = \delta(O) \} \]

If \( V \) is an open set, we can choose an open \( O \subset V \) such that \( \delta(O) = \min\{ \delta(O'): O' \text{ is open and } O' \subset V \} \). Then \( O \in \mathcal{B} \) so \( \mathcal{B} \) is a \( \pi \)-base for \( X \). Let \( \mathcal{M} \) be a maximal collection of pairwise disjoint elements of \( \mathcal{B} \).

Case 1. \( |\mathcal{M}| = \kappa \). Suppose \( D \) is dense in \( X \). We show \( d(D) \geq \kappa \). Let \( S \subset D \) be dense. Then \( S \) is dense in \( X \), thus \( S \cap M \neq \emptyset \) for all \( M \in \mathcal{M} \), so \( |S| \geq \kappa \). Therefore, \( \delta(D) = \kappa \) (since \( \delta(X) = \kappa \)).

Case 2a. \( |\mathcal{M}| < \kappa \), but for all \( M \in \mathcal{M} \), \( \delta(M) < \kappa \). There cannot be a cardinal \( \lambda < \kappa \) s.t. \( \delta(M) \leq \lambda \) for all \( M \in \mathcal{M} \), since if there were, suppose \( D \) is a dense subset of \( X \). Then for each \( M \in \mathcal{M} \) there is \( D_M \subset D \cap M \) which is dense in \( D \cap M \) such that \( |D_M| \leq \delta(M) \leq \lambda \). Then \( \bigcup_{M \in \mathcal{M}} D_M \) is dense in \( D \), since \( \mathcal{M} \) was maximal and \( B \) was a \( \pi \)-base. However, \( \bigcup_{M \in \mathcal{M}} D_M \leq \lambda \cdot |\mathcal{M}| \). This implies that \( \delta(X) \leq \lambda \cdot |\mathcal{M}| < \kappa \), so there can be no such \( \lambda \). Thus there is a sequence \( \langle \kappa_i : i \in \omega \rangle \) converging to \( \kappa \) and a sequence \( \langle M_i : i \in \omega \rangle \) with \( M_i \in \mathcal{M} \) and \( \delta(M_i) > \kappa_i \) for all \( i \). Let

\[ \mathcal{M}' = \{ M_i : i \in \omega \} \cup \{ \bigcup \{ M \in \mathcal{M}: M \neq M_i \text{ for all } i \in \omega \} \} \]

\( \mathcal{M}' \) is a maximal pairwise disjoint collection of open sets in \( X \). For each \( i \), choose a set \( D_i \subset M_i \) such that \( d(D_i) > \kappa_i \) and \( D_i \) is dense in \( M_i \). Then \( D = \bigcup_{i \in \omega} D_i \cup \bigcup \{ M \in \mathcal{M}: M \neq M_i \text{ for all } i \in \omega \} \) is a dense subset of \( X \). Suppose \( D' \) is a dense subset of \( D \). Then \( D' \cap D_i \) is dense in \( D_i \), thus \( |D' \cap D_i| > \kappa_i \). Since the collection \( \{ D_i : i \in \omega \} \) is pairwise disjoint, \( |D'| > |\bigcup_{i \in \omega} D' \cap D_i| = \kappa \). Thus \( d(D) = \kappa \) (since \( \delta(X) = \kappa \), \( d(D) \leq \kappa \)).

Case 2b. There is \( M \in \mathcal{M} \) s.t. \( \delta(M) = \kappa \) (note that since \( \delta(0) \leq \delta(X) \) for all open \( O \subset X \), we cannot have \( \delta(M) > \kappa \)).

Since \( X \) is Hausdorff, we can choose a countable maximal collection \( \{ M_i : i \in \omega \} \) of pairwise disjoint open subsets of \( X \). By the definition of \( \mathcal{M} \), \( \delta(M_i) = \kappa \) for all \( i \). Choose a sequence \( \langle \kappa_i : i \in \omega \rangle \) of cardinals converging to \( \kappa \) with \( \kappa_i < \kappa \) for each \( i \). Choose a dense \( D_i \subset M_i \) s.t. \( d(D_i) > \kappa_i \). Let \( D = \bigcup_{i \in \omega} D_i \cup (X - M) \). By an argument similar to Case 2a, \( d(D) = \kappa \).

It was in this last argument that we needed to know that \( \text{cf}(\kappa) = \omega \), since we could only guarantee that we could choose a countable collection of pairwise disjoint open subsets of \( M \).

When SUP = MAX fails. Suppose \( \kappa \) is a limit cardinal, but not a strong limit, and \( \text{cf}(\kappa) > \omega \). We will construct a space \( X \subset 2^\kappa \) such that \( \delta(X) = \kappa \), but for all dense \( D \subset X \), \( d(D) < \kappa \).
Choose \( \lambda < \kappa \) such that \( 2^\lambda \geq \kappa \). It is well known that \( 2^\kappa \) has a dense subset \( S \) with \( |S| = \lambda \).

Let \( \langle \kappa_\alpha: \alpha < \text{cf}(\kappa) \rangle \) be an increasing sequence of cardinals converging to \( \kappa \) with \( \kappa_0 = 0 \) and \( \kappa_1 = \lambda \). For each \( \alpha < \text{cf}(\kappa) \), let \( \hat{\alpha} = [\kappa_\alpha, \kappa_{\alpha+1}) \). If \( \beta < \kappa \), Let \( \alpha(\beta) \) be the unique \( \alpha < \text{cf}(\kappa) \) such that \( \beta \in \hat{\alpha} \), and if \( J \subseteq \kappa \), let \( \alpha(J) = \{ \alpha(\beta): \beta \in J \} \).

For \( \alpha < \text{cf}(\kappa) \) define

\[
X_\alpha = \{ p \in 2^\kappa: p \mid \hat{\alpha} \subseteq \sigma(\hat{\alpha}) \text{ and there is } s \in S \text{ such that } p \mid (\pi - \hat{\alpha}) = s \mid (\kappa - \hat{\alpha}) \}.
\]

Let \( X = \bigcup_{\alpha < \text{cf}(\kappa)} X_\alpha \). Since \( S \) is dense in \( 2^\kappa \) and \( \sigma(\hat{\alpha}) \) is dense in \( 2^\hat{\alpha} \), \( X_\alpha \) is dense in \( X \) for each \( \alpha \). Also (since \( d(\sigma(\hat{\alpha})) = \kappa_{\alpha+1} \geq \lambda \) \( d(X_\alpha) = \kappa_{\alpha+1} \)). Thus \( \delta(X) \geq \kappa \), and, since \( w(X) = w(2^\kappa) = \kappa \), \( \delta(X) = \kappa \).

Suppose \( D \) is a dense subset of \( X \). We must show that \( d(D) < \kappa \). Note in what follows, that, since \( X \) is dense in \( 2^\kappa \), if \( O \) is an open subset of \( 2^\kappa \), \( O \cap D \neq \emptyset \) if and only if \( O \neq \emptyset \).

Suppose \( h \in H(\kappa) \). We will say \( h \) is good if there is \( \beta < \text{cf}(\kappa) \) s.t. \( \langle h \rangle \cap D \cap X_\beta \neq \emptyset \) and \( \beta \in \alpha(\text{dom}(h)) \). Otherwise, we will say \( h \) is bad. (Of course, whether \( h \) is good or bad depends upon \( D \).

For each \( s \in S \), let

\[
A_s = \{ \beta: \exists p \in X_\beta \cap D \text{ such that } p \mid (\kappa - \hat{\beta}) = s \mid (\kappa - \hat{\beta}) \}.
\]

If \( A_s \) is finite, let \( B_s = A_s \). If \( A_s \) is infinite, let \( B_s \) be a countably infinite subset of \( A_s \).

For each \( s \in S \) and \( \beta \in B_s \), choose \( p(s, \beta) \in D \cap X_\beta \) such that \( p(s, \beta) \mid (\kappa - \hat{\beta}) \) (i.e. \( p(s, \beta) \) is a witness to \( \beta \in B_s \)). Let \( D_G = \{ p(s, \beta): s \in S, \beta \in B_s \} \). Then \( |D_G| \leq \lambda \).

If \( h \in H(\kappa) \) is bad, let \( D_h = \{ \beta < \text{cf}(\kappa): D \cap X_\beta \cap \langle h \rangle \neq \emptyset \} \). Then, by the definition of “bad”, \( D_h \subseteq \alpha(\text{dom}(h)) \). Let \( \mathcal{H} \subseteq H(\kappa) \) be a maximal collection such that if \( h \in \mathcal{H} \), then \( h \) is bad and if \( h_1, h_2 \in \mathcal{H} \) and \( h_1 \neq h_2 \), then \( \langle h_1 \rangle \cap \langle h_2 \rangle = \emptyset \). Since \( c(2^\kappa) = \omega \), \( |\mathcal{H}| \leq \omega \). Let \( J = \bigcup\{ D_h: h \in \mathcal{H} \} \). Since \( D_h \) is finite for each \( h \in \mathcal{H} \), \( |J| \leq \omega \). Finally, let \( D_B = \bigcup\{ D \subseteq X_\beta: \beta \in J \} \). Since \( |X_\beta| < \lambda \cdot |\beta| = \kappa_{\beta+1} < \kappa \), and \( |J| \leq \omega < \text{cf}(\kappa) \), it follows that \( |D_B| < \kappa \).

We can now show that \( D_G \cup D_B \) is a dense subset of \( D \). Suppose \( h \in H(\kappa) \). If \( h \) is good, then there is \( \beta < \text{cf}(\kappa) \) and \( p \in \langle h \rangle \cap D \cap X_\beta \) such that \( \beta \not\in \alpha(\text{dom}(h)) \). Choose \( s \in S \) such that \( p \mid (\kappa - \hat{\beta}) = s \mid (\kappa - \hat{\beta}) \). Then \( s \in \langle h \rangle \). If \( A_s \) is finite (and thus \( B_s = A_s \)), let \( \beta' = \beta \). If \( B_s \) is infinite, choose \( \beta' \in B_s - \alpha(\text{dom}(h)) \). Either way, \( \beta' \in B_s \). Since \( p(s, \beta') \mid (\kappa - \hat{\beta'}) = s \mid (\kappa - \hat{\beta'}) \) and \( s \in \langle h \rangle \), then \( p(s, \beta') \in \langle h \rangle \cap D_G \).

If \( h \) is bad, then there is \( h' \in \mathcal{H} \) such that \( \langle h \rangle \cap \langle h' \rangle \neq \emptyset \). Let \( D \cap X_\beta \cap \langle h \rangle \cap \langle h' \rangle \neq \emptyset \). Then \( \beta \in D_{h'} \subseteq \text{dom}(h) \), so \( D_B \cap \langle h \rangle \neq \emptyset \). Thus \( D_G \cup D_B \) is a dense subset of \( D \). Since \( |D_G| < \lambda < \kappa \), and \( |D_B| < \kappa \), \( d(D) < \kappa \).

4. Questions about compact spaces. For any space \( X \), \( \delta(X) \leq \pi(X) \). It is shown in [J, Theorem 3.14c] that if \( X \) is compact, then \( X \) has a dense left separated sequence of order type \( \pi(X) \). If \( \pi(X) \) is regular, then this sequence has density \( \pi(X) \), so we
have shown that if $X$ is compact and $\pi(X)$ is regular, then $\delta(X) = \pi(X)$, and $\text{SUP} = \text{MAX}$ holds for $\delta$. This raises the following two questions:

(a) If $X$ is compact, does $\delta(X) = \pi(X)$?
(b) If $X$ is compact, does $\text{SUP} = \text{MAX}$ hold for $\delta$?

REFERENCES


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