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THE SUP = MAX PROBLEM FOR δ

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ABSTRACT. Let $\delta(X) = \sup\{d(D)\colon D$ is a dense subspace of $X\}$. It is shown that if κ is a limit cardinal, but not a strong limit, and $\mathrm{cf}(\kappa) > \omega$, then there is a 0-dimensional Hausdorff space X such that $\delta(X) = \kappa$, but for all dense $D \subset X$, $d(D) < \kappa$. For all other values of κ, if X is Hausdorff and $\delta(X) = \kappa$, then there is a dense $D \subset X$ such that $d(D) = \kappa$.

1. Introduction. We consider the SUP = MAX problem for the cardinal function δ defined as

$$\delta(X) = \sup\{d(D)\colon D \text{ is a dense subspace of } X\}.$$

For Hausdorff spaces, the solution is given by Theorem 1.

It is easy to verify that $d(X) \leqslant \delta(X) \leqslant d(X) \cdot t(X)$. Let $X = 2^{\omega_1}$. Then $d(X) = \omega$. However, $\Sigma(2^{\omega_1}) \subset X$ is dense, and $d(\Sigma(2^{\omega_1})) = \omega_1 = w(X)$. Thus we have an example where $\delta(X) > d(X)$.

THEOREM 1. *If κ is a limit cardinal, but not a strong limit, and* $\mathrm{cf}(\kappa) > \omega$, *then there is a 0-dimensional Hausdorff space X such that $\delta(X) = \kappa$, but for all dense $D \subset X$, $d(D) < \kappa$. Otherwise, if X is Hausdorff and $\delta(X) = \kappa$, then there is a dense $D \subset X$ such that $d(D) = \kappa$.*

We will prove Theorem 1 in §§2 and 3.

As always with the SUP = MAX problem, we need only consider the case where $\delta(X) = \kappa$ is a limit. It is easy to see that the theorem fails for non-Hausdorff X. Suppose, for example, $\kappa = \bigcup_{\alpha < \mathrm{cf}(\kappa)} \kappa_\alpha$. Let $\{X_\alpha\colon \alpha < \mathrm{cf}(\kappa)\}$ be a pairwise disjoint collection of sets with $|X_\alpha| = \kappa_\alpha$. Let $X = \bigcup_{\alpha < \mathrm{cf}(\kappa)} X_\alpha$. Define a set $O \subset X$ to be open if either $O = \varnothing$ or $|X_\alpha - O| < \kappa_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$. X is T_1 but not T_2. Since X_α is dense in X, $\delta(X) = \kappa$. If $D \subset X$ is dense, then $|D \cap X_\alpha| = \kappa_\alpha$ for some $\alpha < \mathrm{cf}(\kappa)$ (otherwise D is closed), and then $D \cap X_\alpha$ is dense, so $d(D) < \kappa$. Thus SUP = MAX fails for all limits.

We will use the following notation. If S is a set, $\sigma(S) = \{p \in 2^S\colon |p^{\leftarrow}(1)| < \omega\}$. Note that $\sigma(S)$ is dense in 2^S. If S is a set, $H(S)$ is the collection of all finite partial functions from S into $\{0,1\}$. If $h \in H(S)$, then $\langle h \rangle = \{p \in 2^S\colon p \text{ extends } h\}$. Thus $\{\langle h \rangle\colon h \in H(S)\}$ is the standard basis for 2^S.

For the rest of the paper, we will assume that all spaces are Hausdorff.

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2. When SUP = MAX. We first prove the second part of the theorem. As noted, we may assume κ is a limit cardinal. Suppose $\delta(X) = \kappa$ and κ is a strong limit (i.e. if $\lambda < \kappa$ then $2^\lambda < \kappa$). Then $d(X) = \kappa$ since $|X| \leqslant \exp(\exp(d(X)))$ [**1**, Theorem 2.4] and $\delta(X) \leqslant |X|$.

Suppose $\delta(X) = \kappa$ and $\text{cf}(\kappa) = \omega$. Let

$$\mathscr{B} = \{ O \subset X\colon O \neq \varnothing \text{ is open and if } U \subset O \text{ is open then } \delta(U) = \delta(O) \}.$$

If V is an open set, we can choose an open $O \subset V$ such that $\delta(O) = \min\{\delta(O')\colon O'$ is open and $O' \subset V\}$. Then $O \in \mathscr{B}$ so $\mathscr{B}$ is a π-base for X. Let $\mathscr{M}$ be a maximal collection of pairwise disjoint elements of $\mathscr{B}$.

Case 1. $|\mathscr{M}| = \kappa$. Suppose D is dense in X. We show $d(D) \geqslant \kappa$. Let $S \subset D$ be dense. Then S is dense in X, thus $S \cap M \neq \varnothing$ for all $M \in \mathscr{M}$, so $|S| \geqslant \kappa$. Therefore, $d(D) = \kappa$ (since $\delta(X) = \kappa$).

Case 2a. $|\mathscr{M}| < \kappa$, but for all $M \in \mathscr{M}$, $\delta(M) < \kappa$. There cannot be a cardinal $\lambda < \kappa$ s.t. $\delta(M) \leqslant \lambda$ for all $M \in \mathscr{M}$, since if there were, suppose D is a dense subset of X. Then for each $M \in \mathscr{M}$ there is $D_M \subset D \cap M$ which is dense in $D \cap M$ such that $|D_M| \leqslant \delta(M) \leqslant \lambda$. Then $\bigcup_{M \in \mathscr{M}} D_M$ is dense in D, since $\mathscr{M}$ was maximal and $\mathscr{B}$ was a π-base. However, $|\bigcup_{M \in \mathscr{M}} D_M| \leqslant \lambda \cdot |\mathscr{M}|$. This implies that $\delta(X) \leqslant \lambda \cdot |\mathscr{M}| < \kappa$, so there can be no such λ. Thus there is a sequence $\langle \kappa_i\colon i \in \omega\rangle$ converging to κ and a sequence $\langle M_i\colon i \in \omega\rangle$with $M_i \in \mathscr{M}$ and $\delta(M_i) > \kappa_i$ for all i. Let

$$\mathscr{M}' = \{ M_i\colon i \in \omega\} \cup \left\{ \bigcup \{ M \in \mathscr{M}\colon M \neq M_i \text{ for all } i \in \omega\} \right\}.$$

$\mathscr{M}'$ is a maximal pairwise disjoint collection of open sets in X. For each i, choose a set $D_i \subset M_i$ such that $d(D_i) > \kappa_i$ and D_i is dense in M_i. Then $D = \bigcup_{i \in \omega} D_i \cup \bigcup\{M \in \mathscr{M}\colon M \neq M_i \text{ for all } i \in \omega\}$ is a dense subset of X. Suppose D' is a dense subset of D. Then $D' \cap D_i$ is dense in D_i, thus $|D' \cap D_i| > \kappa_i$. Since the collection $\{D_i\colon i \in \omega\}$ is pairwise disjoint, $|D'| \geqslant |\bigcup_{i \in \omega} D' \cap D_i| = \kappa$. Thus $d(D) = \kappa$ (since $\delta(X) = \kappa$, $d(D) \leqslant \kappa$).

Case 2b. There is $M \in \mathscr{M}$ s.t. $\delta(M) = \kappa$ (note that since $\delta(0) \leqslant \delta(X)$ for all open $O \subset X$, we cannot have $\delta(M) > \kappa$).

Since X is Hausdorff, we can choose a countable maximal collection $\{M_i\colon i \in \omega\}$ of pairwise disjoint open subsets of M. By the definition of $\mathscr{M}$, $\delta(M_i) = \kappa$ for all i. Choose a sequence $\langle \kappa_i\colon i \in \omega\rangle$ of cardinals converging to κ with $\kappa_i < \kappa$ for each i. Choose a dense $D_i \subset M_i$ s.t. $d(D_i) > \kappa_i$. Let $D = \bigcup_{i \in \omega} D_i \cup (X - M)$. By an argument similar to Case 2a, $d(D) = \kappa$.

It was in this last argument that we needed to know that $\text{cf}(\kappa) = \omega$, since we could only guarantee that we could choose a countable collection of pairwise disjoint open subsets of M.

When SUP = MAX **fails.** Suppose κ is a limit cardinal, but not a strong limit, and $\text{cf}(\kappa) > \omega$. We will construct a space $X \subset 2^\kappa$ such that $\delta(X) = \kappa$, but for all dense $D \subset X$, $d(D) < \kappa$.

Choose $\lambda < \kappa$ such that $2^\lambda \geqslant \kappa$. It is well known that 2^κ has a dense subset S with $|S| = \lambda$.

Let $\langle \kappa_\alpha \colon \alpha < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of cardinals converging to κ with $\kappa_0 = 0$ and $\kappa_1 = \lambda$. For each $\alpha < \mathrm{cf}(\kappa)$, let $\hat{\alpha} = [\kappa_\alpha, \kappa_{\alpha+1})$. If $\beta < \kappa$, Let $\alpha(\beta)$ be the unique $\alpha < \mathrm{cf}(\kappa)$ such that $\beta \in \hat{\alpha}$, and if $J \subset \kappa$, let $\alpha(J) = \{\alpha(\beta) \colon \beta \in J\}$.

For $\alpha < \mathrm{cf}(\kappa)$ define

$$X_\alpha = \{ p \in 2^\kappa \colon p \,|\, \hat{\alpha} \in \sigma(\hat{\alpha}) \text{ and there is } s \in S \text{ such that } p \,|\, (\pi - \hat{\alpha}) = s \,|\, (\kappa - \hat{\alpha})\}.$$

Let $X = \bigcup_{\alpha < \mathrm{cf}(\kappa)} X_\alpha$. Since S is dense in 2^κ and $\sigma(\hat{\alpha})$ is dense in $2^{\hat{\alpha}}$, X_α is dense in X for each α. Also (since $d(\sigma(\hat{\alpha})) = \kappa_{\alpha+1} \geqslant \lambda$) $d(X_\alpha) = \kappa_{\alpha+1}$. Thus $\delta(X) \geqslant \kappa$, and, since $w(X) = w(2^\kappa) = \kappa$, $\delta(X) = \kappa$.

Suppose D is a dense subset of X. We must show that $d(D) < \kappa$. Note in what follows, that, since X is dense in 2^κ, if O is an open subset of 2^κ, $O \cap D \neq \varnothing$ if and only if $O \neq \varnothing$.

Suppose $h \in H(\kappa)$. We will say h is *good* if there is $\beta < \mathrm{cf}(\kappa)$ s.t. $\langle h \rangle \cap D \cap X_\beta \neq \varnothing$ and $\beta \notin \alpha(\mathrm{dom}(h))$. Otherwise, we will say h is *bad*. (Of course, whether h is good or bad depends upon D.)

For each $s \in S$, let

$$A_s = \{\beta \colon \exists p \in X_\beta \cap D \text{ such that } p \,|\, (\kappa - \hat{\beta}) = s \,|\, (\kappa - \hat{\beta})\}.$$

If A_s is finite, let $B_s = A_s$. If A_s is infinite, let B_s be a countably infinite subset of A_s.

For each $s \in S$ and $\beta \in B_s$, choose $p(s, \beta) \in D \cap X_\beta$ such that $p(s,\beta) \,|\, (\kappa - \hat{\beta}) = s \,|\, (\kappa - \hat{\beta})$ (i.e. $p(s, \beta)$ is a witness to $\beta \in B_s$). Let $D_G = \{p(s,\beta) \colon s \in S, \beta \in B_s\}$. Then $|D_G| \leqslant \lambda$.

If $h \in H(\kappa)$ is bad, let $D_h = \{\beta < \mathrm{cf}(\kappa) \colon D \cap X_\beta \cap \langle h \rangle \neq \varnothing\}$. Then, by the definition of "bad", $D_h \subset \alpha(\mathrm{dom}(h))$. Let $\mathscr{J} \subset H(\kappa)$ be a maximal collection such that if $h \in \mathscr{J}$, then h is bad and if $h_1, h_2 \in \mathscr{J}$ and $h_1 \neq h_2$, then $\langle h_1 \rangle \cap \langle h_2 \rangle = \varnothing$. Since $c(2^\kappa) = \omega$, $|\mathscr{J}| \leqslant \omega$. Let $J = \bigcup\{D_h \colon h \in \mathscr{J}\}$. Since D_h is finite for each $h \in \mathscr{J}$, $|J| \leqslant \omega$. Finally, let $D_B = \bigcup\{D \cap X_\beta \colon \beta \in J\}$. Since $|X_\beta| \leqslant \lambda \cdot |\hat{\beta}| = \kappa_{\beta+1} < \kappa$, and $|J| \leqslant \omega < \mathrm{cf}(\kappa)$, it follows that $|D_B| < \kappa$.

We can now show that $D_G \cup D_B$ is a dense subset of D. Suppose $h \in H(\kappa)$. If h is good, then there is $\beta < \mathrm{cf}(\kappa)$ and $p \in \langle h \rangle \cap D \cap X_\beta$ such that $\beta \notin \alpha(\mathrm{dom}(h))$. Choose $s \in S$ such that $p \,|\, (\kappa - \hat{\beta}) = s \,|\, (\kappa - \hat{\beta})$. Then $s \in \langle h \rangle$. If A_s is finite (and thus $B_s = A_s$), let $\beta' = \beta$. If B_s is infinite, choose $\beta' \in B_s - \alpha(\mathrm{dom}(h))$. Either way, $\beta' \in B_s - \alpha(\mathrm{dom}(h))$. Since $p(s, \beta') \,|\, (\kappa - \hat{\beta}') = s \,|\, (\kappa - \hat{\beta}')$ and $s \in \langle h \rangle$, then $p(s, \beta') \in \langle h \rangle \cap D_G$.

If h is bad, then there is $h' \in \mathscr{J}$ such that $\langle h \rangle \cap \langle h' \rangle \neq \varnothing$. Let $D \cap X_\beta \cap \langle h \rangle \cap \langle h' \rangle \neq \varnothing$. Then $\beta \in D_{h'} \subset j$, so $D_B \cap \langle h \rangle \neq \varnothing$. Thus $D_G \cup D_B$ is a dense subset of D. Since $|D_G| \leqslant \lambda < \kappa$, and $|D_B| < \kappa$, $d(D) < \kappa$.

4. Questions about compact spaces. For any space X, $\delta(X) \leqslant \pi(X)$. It is shown in [**J**, Theorem 3.14c] that if X is compact, then X has a dense left separated sequence of order type $\pi(X)$. If $\pi(X)$ is regular, then this sequence has density $\pi(X)$, so we have shown that if X is compact and $\pi(X)$ is regular, then $\delta(X) = \pi(X)$, and SUP = MAX holds for δ. This raises the following two questions:

(a) If X is compact, does $\delta(X) = \pi(X)$?

(b) If X is compact, does SUP = MAX hold for δ?

References

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