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SHORTER NOTES

The purpose of this department is to publish very short papers of unusually polished character, for which there is no other outlet.

ON MINKOWSKI'S SINGULAR FUNCTION

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The function f on the closed unit interval I to be considered here was introduced by Minkowski. It maps the algebraic irrationalities of degree at most 2 continuously to the rationals in I. If $z = a/b$ and $z' = c/d$ are (reduced) fractions, then we write $z * z'$ for the mediant $(a+c)/(b+d)$ and $z \cdot z'$ for the arithmetic mean. Let Q denote the (dense) set of rationals in I with terminating dyadic expansion, i.e. the set of numbers $x_n^\nu = n2^{1-\nu}$ $(n = 0, \ldots, 2^{\nu-1},\ \nu \in N)$. Define a function g on Q inductively as follows: Let $y_0^1 = 0$, $y_1^1 = 1$. Suppose the values $y_n^\nu = g(x_n^\nu)$ have been defined on level ν. Then put $y_{2k}^{\nu+1} = y_k^\nu$ $(0 \le k \le 2^\nu)$ and insert $y_{2k-1}^{\nu+1} = y_{k-1}^\nu * y_k^\nu$ $(1 \le k \le 2^\nu)$ to obtain the images $y_n^{\nu+1} = g(x_n^{\nu+1})$ on level $\nu+1$. The resulting function g is a strictly increasing continuous bijection from Q to the rationals. Note that g is constructed so as to have the property $g(x)*g(x') = g(x \cdot x')$ whenever $x, x' \in Q$, $x' - x = 2^{1-\nu}$, $\nu \in N$. The inverse of g can be extended to the reals to yield a strictly increasing continuous distribution function f inducing a (singular) probability measure on I.

We will be concerned with the regular and semiregular continued fraction expansions $y = \underline{1}\rceil\overline{\alpha} + \underline{1}\rceil\overline{\beta} + \cdots = [\alpha, \beta, \ldots]$, $\alpha, \beta, \ldots \in N$ and $y = \underline{1}\rceil\overline{a} - \underline{1}\rceil\overline{b} - \cdots = [\![a, b, \ldots]\!]$, $a, b, \ldots \in N \setminus \{1\}$. In the dyadic expansion of elements $x \in I$ we collect 0's and 1's and write $x = {}^{\cdot}0_{\alpha-1}1_\beta 0_\gamma 1_\delta 0_\varepsilon \cdots$, letting $\alpha - 1, \beta, \gamma, \delta, \varepsilon, \ldots$ denote string lengths $(\alpha, \beta, \gamma, \delta, \varepsilon, \ldots \in N)$. The following has been proved by different arguments (see [**2**] for references):

(i) $[\![\alpha+1; 2_{\beta-1}, \gamma+2, 2_{\delta-1}, \varepsilon+2, \ldots]\!] = f^{-1}({}^{\cdot}0_{\alpha-1}1_\beta 0_\gamma 1_\delta 0_\varepsilon \cdots)$ (this is [**2**, (5)]);

(ii) f is singular.

Our purpose here is to present a simple unified approach to both properties by splitting up (i) into the following equalities, the second one making (ii) evident:

$$[\![\alpha + 1; 2_{\beta-1}, \gamma + 2, 2_{\delta-1}, \varepsilon + 2, \ldots]\!] = [\alpha, \beta, \gamma, \delta, \varepsilon, \ldots]; \tag{1}$$

$$[\alpha, \beta, \gamma, \delta, \varepsilon, \ldots] = f^{-1}({}^{\cdot}0_{\alpha-1}1_\beta 0_\gamma 1_\delta 0_\varepsilon \cdots). \tag{2}$$

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PROOF. We obtain (1) by iterating the obvious identity $[\![m+1, 2_{k-1}, z]\!] = [m, k, z-1]$ $(z > 1, k \in N$; we allow 2-strings to be empty). By the continuity of f^{-1} it suffices to prove (2) for arguments from the dense set Q. Define a function h on Q as follows: if $x \in Q$ has expansion $\dot{}0_{\alpha-1}1_\beta \cdots 0_\phi 1_\psi$, say, then put $h(x) = [\alpha, \beta, \ldots, \phi, \psi]$. We claim that h and g coincide on Q. Clearly this will be proved if we can show that h shares the isomorphism property of g mentioned. Let x, x' be any elements from Q with $x' - x = 2^{1-\nu}$, $\nu \in N$. Suppose $x = n2^{1-\nu} = \dot{}0_{\alpha-1}1_\beta \cdots 0_\phi 1_\psi$.

Case 1. n has dyadic expansion $n = 1_\beta \cdots 0_\phi 1_\psi 0\dot{}_\omega$, $\omega \geq 1$. Then

$$\begin{aligned} h(x) * h(x') &= h(\dot{}0_{\alpha-1} \cdots 0_\phi 1_\psi) * h(\dot{}0_{\alpha-1} \cdots 0_\phi 1_\psi 0_{\omega-1} 1) \\ &= [\alpha, \ldots, \phi, \psi] * [\alpha, \ldots, \phi, \psi, \omega - 1, 1] = [\alpha, \ldots, \psi, \omega, 1] \\ &= h(\dot{}0_{\alpha-1} \cdots 0_\phi 1_\psi 0_\omega 1) = h(x \cdot x'). \end{aligned}$$

Case 2. $n = 1_\beta \cdots 0_\phi 1\dot{}_\psi$. Here

$$\begin{aligned} h(x) * h(x') &= h(\dot{}0_{\alpha-1} \cdots 0_\phi 1_\psi) * h(\dot{}0_{\alpha-1} \cdots 0_{\phi-1} 1) \\ &= [\alpha, \ldots, \phi, \psi] * [\alpha, \ldots, \phi - 1, 1] = [\alpha, \ldots, \phi, \psi + 1] \\ &= h(\dot{}0_{\alpha-1} \cdots 0_\phi 1_{\psi+1}) = h(x \cdot x'), \end{aligned}$$

as required. Our alternative proof of (i) is complete.

Finally, let Y denote the subset of I consisting of all elements $y \in I$ for which the frequency of each $k \in N$ in the sequence of partial denominators in the regular expansion $y = [\alpha_1(y), \alpha_2(y), \ldots]$ is $\log(1 + 1/(k^2 + 2k))/\log 2$. It is well known [**1**] that Y has (full) Lebesgue measure 1. But by the very nature of the digital correspondence (2) the image set $f(Y)$ must be a null-set, as the frequencies of $k \in N$ in the sequence of lengths of 0-strings and 1-strings in the dyadic expansion clearly differ from the above values almost everywhere on I. This proves (ii).

REFERENCES

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