SHORTER NOTES

The purpose of this department is to publish very short papers of unusually polished character, for which there is no other outlet.

ON MINKOWSKI’S SINGULAR FUNCTION

G. RAMHARTER

The function $f$ on the closed unit interval $I$ to be considered here was introduced by Minkowski. It maps the algebraic irrationalities of degree at most 2 continuously to the rationals in $I$. If $z = a/b$ and $z' = c/d$ are (reduced) fractions, then we write $z * z'$ for the median $(a + c)/(b + d)$ and $z \cdot z'$ for the arithmetic mean. Let $Q$ denote the (dense) set of rationals in $I$ with terminating dyadic expansion, i.e. the set of numbers $x_n = n2^{-n}$ ($n = 0, \ldots, 2^{v-1}$, $\nu \in N$). Define a function $g$ on $Q$ inductively as follows: Let $y^0_0 = 0, y^1_1 = 1$. Suppose the values $y^{\nu}_n = g(x^\nu_n)$ have been defined on level $\nu$. Then put $y^{\nu+1}_{2k} = y^\nu_k$ ($0 \leq k \leq 2^\nu$) and insert $y^{\nu+1}_{2k-1} = y^{\nu}_k * y^\nu_k$ ($1 \leq k \leq 2^\nu$) to obtain the images $y^{\nu+1}_n = g(x^\nu_{n+1})$ on level $\nu+1$. The resulting function $g$ is a strictly increasing continuous bijection from $Q$ to the rationals. Note that $g$ is constructed so as to have the property $g(x) * g(x') = g(x \cdot x')$ whenever $x, x' \in Q$, $x' - x = 2^{1-\nu}$, $\nu \in N$. The inverse of $g$ can be extended to the reals to yield a strictly increasing continuous distribution function $f$ inducing a (singular) probability measure on $I$.

We will be concerned with the regular and semiregular continued fraction expansions $\alpha = [\alpha, \beta, \ldots, \gamma, \delta, \epsilon, \ldots]$ and $\alpha = [\alpha, \beta, \gamma, \delta, \epsilon, \ldots]$ of elements $x \in I$ we collect 0's and 1's and write $x = 0_{\alpha-1}1_{\beta}0_{\gamma}1_{\delta}0_{\epsilon} \cdots$, letting $\alpha - 1, \beta, \gamma, \delta, \epsilon, \ldots$ denote string lengths ($\alpha, \beta, \gamma, \delta, \epsilon, \ldots \in N$). The following has been proved by different arguments [2] for references:

(i) $[\alpha + 1; 2_{\beta-1}, \gamma + 2, 2_{\delta-1}, \epsilon + 2, \ldots] = f^{-1}(0_{\alpha-1}1_{\beta}0_{\gamma}1_{\delta}0_{\epsilon} \cdots)$ (this is [2, (5)]);
(ii) $f$ is singular.

Our purpose here is to present a simple unified approach to both properties by splitting up (i) into the following equalities, the second one making (ii) evident:

1. $[\alpha + 1; 2_{\beta-1}, \gamma + 2, 2_{\delta-1}, \epsilon + 2, \ldots] = [\alpha, \beta, \gamma, \delta, \epsilon, \ldots]$;
2. $[\alpha, \beta, \gamma, \delta, \epsilon, \ldots] = f^{-1}(0_{\alpha-1}1_{\beta}0_{\gamma}1_{\delta}0_{\epsilon} \cdots)$. 

Received by the editors April 23, 1986.

Proof. We obtain (1) by iterating the obvious identity \([m + 1, 2k_1, z] = [m, k, z - 1]\) \((z > 1, k \in \mathbb{N};\) we allow 2-strings to be empty). By the continuity of \(f^{-1}\) it suffices to prove (2) for arguments from the dense set \(Q\). Define a function \(h\) on \(Q\) as follows: if \(x \in Q\) has expansion \(0_{\alpha-1}1_{\beta} \cdots 0_{\phi}1_{\psi}\), say, then put \(h(x) = [\alpha, \beta, \ldots, \phi, \psi]\). We claim that \(h\) and \(g\) coincide on \(Q\). Clearly this will be proved if we can show that \(h\) shares the isomorphism property of \(g\) mentioned. Let \(x, x'\) be any elements from \(Q\) with \(x' - x = 2^{1-\nu}\), \(\nu \in \mathbb{N}\). Suppose \(x = n2^{1-\nu} = 0_{\alpha-1}1_{\beta} \cdots 0_{\phi}1_{\psi}\).

Case 1. \(n\) has dyadic expansion \(n = 1_{\beta} \cdots 0_{\phi}1_{\psi}0_\omega\), \(\omega \geq 1\). Then
\[
h(x) * h(x') = h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi}) * h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi}0_{\omega-1})
\]
\[
= [\alpha, \ldots, \phi, \psi] * [\alpha, \ldots, \phi, \psi, \omega - 1, 1] = [\alpha, \ldots, \psi, \omega, 1]
\]
\[
= h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi}0_\omega) = h(x \cdot x').
\]

Case 2. \(n = 1_{\beta} \cdots 0_{\phi}1_{\psi}\): Here
\[
h(x) * h(x') = h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi}) * h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi})
\]
\[
= [\alpha, \ldots, \phi, \psi] * [\alpha, \ldots, \phi - 1, 1] = [\alpha, \ldots, \phi, \psi + 1]
\]
\[
= h(0_{\alpha-1} \cdots 0_{\phi}1_{\psi + 1}) = h(x \cdot x'),
\]
as required. Our alternative proof of (i) is complete.

Finally, let \(Y\) denote the subset of \(I\) consisting of all elements \(y \in I\) for which the frequency of each \(k \in \mathbb{N}\) in the sequence of partial denominators in the regular expansion \(y = [\alpha_1(y), \alpha_2(y), \ldots]\) is \(\log(1 + 1/(k^2 + 2k))/\log 2\). It is well known [1] that \(Y\) has (full) Lebesgue measure 1. But by the very nature of the digital correspondence (2) the image set \(f(Y)\) must be a null-set, as the frequencies of \(k \in \mathbb{N}\) in the sequence of lengths of 0-strings and 1-strings in the dyadic expansion clearly differ from the above values almost everywhere on \(I\). This proves (ii).

References


Wiedner Hauptstrasse 8-10/114, A-1040 Vienna, Austria