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BEST CONSTANT FOR THE RATIO OF THE FIRST TWO EIGENVALUES OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH POSITIVE POTENTIALS

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ABSTRACT. We prove the optimal upper bound $\lambda_2/\lambda_1 \leq 4$ for the ratio of the first two eigenvalues of one-dimensional Schrödinger operators with nonnegative potentials. Equality holds if and only if the potential vanishes identically.

In this note we improve upon the one-dimensional portion of a theorem of Singer, Wong, Yau, and Yau [**7**] concerning the first two eigenvalues of a Schrödinger operator on a bounded domain with Dirichlet boundary conditions. Our result is best possible for the class of nonnegative potentials; in addition, it is an easy consequence of a commutation formula and the Rayleigh-Ritz inequality.

THEOREM. *Let $[a,b]$ be a finite interval and let V be a nonnegative real-valued function in $L^1(a,b)$. Let $H = -d^2/dx^2 + V(x)$ with Dirichlet boundary conditions act on $L^2(a,b)$, and let $\lambda_1[V]$ and $\lambda_2[V]$ denote the first two eigenvalues of H. Then $\lambda_2[V]/\lambda_1[V] \leq 4$ with equality if and only if $V = 0$.*

PROOF. Let u_1 be the groundstate eigenfunction of H, i.e. u_1 is in the domain of H and $Hu_1 = \lambda_1 u_1$. By the commutation formula (see, in particular, [**3**] and also [**1**, **2**, **5**]), the operator $\tilde{H} = -d^2/dx^2 + V - 2(u_1'/u_1)'$ has the same spectrum as H except for λ_1. Thus we can obtain an upper bound for λ_2 by using the Rayleigh-Ritz inequality on $\tilde{H}$. Taking u_1^2 as our trial function we find $\tilde{H}u_1^2 = -4u_1u_1'' + Vu_1^2 = 4\lambda_1 u_1^2 - 3Vu_1^2$ and hence

$$\lambda_2 \leq 4\lambda_1 - 3\frac{\int_a^b V u_1^4\,dx}{\int_a^b u_1^4\,dx}. \tag{1}$$

This shows that $\lambda_2/\lambda_1 \leq 4$ for nonnegative potentials. Moreover, equality holds if and only if $V = 0$ since u_1 is positive and continuous on (a,b). □

REMARKS. 1. The choice of u_1^2 as the trial function was motivated by the fact that u_1^2 is the groundstate wavefunction of $\tilde{H}$ when $V = 0$.

2. H above is to be interpreted in the sense of quadratic forms following [**3**], for example.

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3. Our result reads $\lambda_2 - \lambda_1 \leq 3\lambda_1$ when stated as a result about $\lambda_2 - \lambda_1$. This improves upon the bound $\lambda_2 - \lambda_1 \leq 4\lambda_1$ found in [**7** and **1**]. Such a bound (for general dimension) had also been obtained by Harrell [**4**] based on the work of Payne, Pólya, and Weinberger [**6**]. The paper [**1**] derived the bound $\lambda_2 - \lambda_1 \leq 4\lambda_1$ using the commutation formula and the Rayleigh-Ritz inequality in much the way we do above but with u_1 rather than u_1^2 as trial function.

4. We note that formula (1) holds for all V, not just for nonnegative V. In addition, we would expect formula (1) and the inequality $\lambda_2/\lambda_1 \leq 4$ to hold for singular potentials ($V \in L^1_{\text{loc}}(a, b)$, say) and also for problems on the line and half-line where $V(x) \to \infty$ as $x \to \pm\infty$. Of course, we always assume that Dirichlet boundary conditions are imposed at limit circle endpoints. Formally, all these results follow by commutation but one needs to take care that the operator $\tilde{H}$ generated in this way is well defined, selfadjoint, and has Dirichlet boundary conditions (when needed).

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NOTE ADDED IN PROOF. In connection with our remark 4, it has been pointed out to us by Barry Simon that extension of the bound $\lambda_2/\lambda_1 \leq 4$ to the line and half-line with $V(x) \to \infty$ as $|x| \to \infty$ follows from the given case by a limiting argument; in particular, the operator obtained by inserting infinite walls at $\pm a$ or 0 and a (infinite walls = Dirichlet boundary conditions) converges to the original operator in norm resolvent sense as $a \to \infty$ and this guarantees convergence of the eigenvalues. Since the bound holds for all finite-domain problems with Dirichlet boundary conditions it must continue to hold in the limit.

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