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A WILD CANTOR SET AS THE LIMIT SET OF A CONFORMAL GROUP ACTION ON S^3

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ABSTRACT. We construct a conformal action of a free group of finite rank on S^3 whose set of discontinuity Λ is a wild Cantor set.

1. Introduction. This paper is a result of independent efforts by the authors to answer a question asked by M. Freedman and R. Skora in their recent manuscript [**FS**].

There they construct an exotic example of quasiconformal group action on S^3 whose limit set is a wild Cantor set. Some interesting features of that example are that each group element is topologically conjugate to a loxodromic conformal diffeomorphism of S^3, but the entire group is *not* topologically conjugate to a conformal group.

Below we construct a *conformal* group G (abstractly a free group of finite rank) whose limit set is a wild Cantor set. As opposed to the Freedman-Skora example, the group G will contain lots of parabolic conformal diffeomorphisms.

Consequently, it has been conjectured that a wild Cantor set cannot be the limit set of a conformal free group action with no parabolics.

For the sake of brevity, we abuse notation slightly by writing $\bigcup \mathcal{C}$ for $\bigcup_{C\in\mathcal{C}} C$, where $\mathcal{C}$ is a finite collection of subsets of $\mathbf{R}^3$ or S^3. We define

$$\operatorname{mesh}\mathcal{C} = \max\{\operatorname{diam} C | C \in \mathcal{C}\}$$

where metrics on $\mathbf{R}^3$ and S^3 are standard.

2. The example. Our example is modeled on the standard Schottky action (cf. [**Ch**, **Ma**]), except that we allow defining spheres to touch.

By a *pair of eyeglasses* we mean the compactum E consisting of two disjoint simple closed curves S_1, S_2 joined by an arc A. Consider the embedding of E in $\mathbf{R}^3$ defined by Figure 1 (the Hopf link plus an arc joining the components).

Now let $\mathcal{C}$ be a collection of small round balls placed along $E \subset \mathbf{R}^3$ so that adjacent balls touch (see Figure 2). Note that most elements of $\mathcal{C}$ will have two points of contact. There are two exceptional elements T_1, T_2 that contain nonmanifold points of E; they have three points of contact. Note that balls along the circular parts S_1, S_2 of E do not separate between their contact points, while those along A separate between their contact points.

Let $\varphi\colon \mathcal{C} \to \mathcal{C}$ be a fixed point-free involution ($\varphi \cdot \varphi = \mathrm{Id}$) such that

(i) $\varphi(T_1) = T_2$; and

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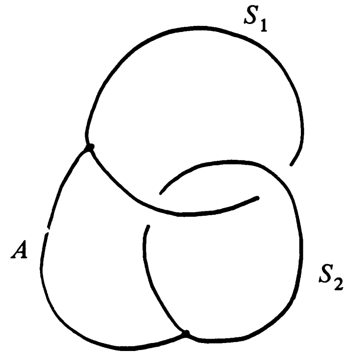

FIGURE 1

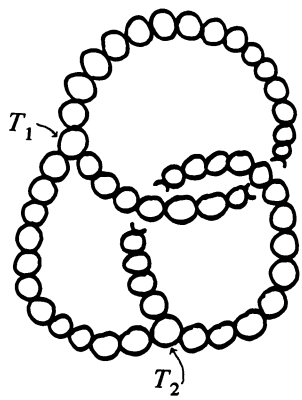

FIGURE 2

(ii) along each circular part S_1, S_2 of E there are at least two balls C', C'' such that $\varphi(C'), \varphi(C'')$ lie along A.

The collection $\mathcal{C}$ can be transferred via the inverse of the stereographic projection to S^3. Suppressing the stereographic projection, we have $E \subset S^3$, $\mathcal{C} =$ collection of round balls in S^3. For each $C \in \mathcal{C}$ choose a conformal diffeomorphism $h_C \colon S^3 \to S^3$ so that

(iii) $h_C(C) = S^3 - \operatorname{int} \varphi(C)$;

(iv) h_C maps the points of contact of C to the points of contact of $\varphi(C)$; and

(v) $h_{\varphi(C)} = h_C^{-1}$.

Let G be the group of conformal diffeomorphisms of S^3 generated by $\{h_C | C \in \mathcal{C}\}$. If the collection $\mathcal{C}$ consisted of disjoint balls, G would be the classical Schottky action, whose limit set is a tame Cantor set. The same arguments apply to our case to show that

(1) G is (abstractly) a free group of finite rank,

(2) G acts freely and properly discontinously in the complement of its limit set Λ,

(3) $\Lambda = \bigcap_{n=0}^{\infty}(\bigcup \mathcal{C}_n)$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_{n+1} = \{h_C(C_n) | C \in \mathcal{C}, C_n \in \mathcal{C}_n, h_C(C_n)$ is contained in an element of $\mathcal{C}_n\}$. Also, $\operatorname{mesh} \mathcal{C}_n \to 0$ as $n \to \infty$.

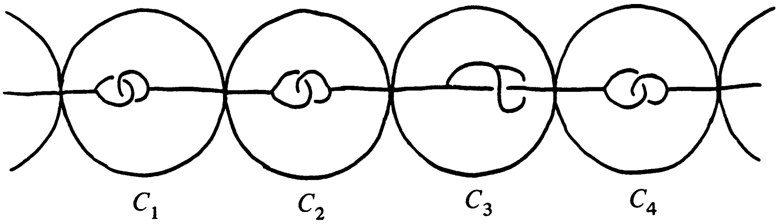

FIGURE 3

In Figure 3 we have drawn a part of the "core" of $\bigcup \mathcal{C}_1$. In the picture, we assumed that $\varphi(C_1), \varphi(C_2), \varphi(C_4)$ lie along the arc A, while $\varphi(C_3)$ lies along a circular part of E.

The main result of this paper is

THEOREM. *Λ is a wild Cantor set.*

PROOF. We first show that Λ is totally disconnected, i.e. that it does not contain any nondegenerate continuum K. Assuming $\operatorname{diam} K > \varepsilon > 0$, choose $n \in \mathbf{N}$ so that $\operatorname{mesh} \mathcal{C}_n < \varepsilon/2$. Since $K \subseteq \Lambda \subseteq \bigcup \mathcal{C}_n$, it follows that there is $C_n \in \mathcal{C}_n$ such that $K \cap C_n$ spans between two different contact points of C_n. Find (unique) $g \in G$ (which will be a word in h_C's of length n) and $C_0 \in \mathcal{C}_0 = \mathcal{C}$ such that $g(C_0) = C_n$. Then $K' = g^{-1}(K \cap C_n)$ is a compactum in $C_0 \cap \Lambda$ that spans between two different contact points of C_0.

Since $K' \subseteq C_0 \cap (\bigcup \mathcal{C}_1)$, it follows that $h_{C_0}(K') \subseteq (S^3 - \operatorname{int} \varphi(C_0)) \cap (\bigcup \mathcal{C}_0)$. If $\varphi(C_0)$ lies along A, then the latter set does not span between contact points of $\varphi(C_0)$, while $h_{C_0}(K')$ does, a contradiction. If $\varphi(C_0)$ lies along a circular part S_i of E, then $h_{C_0}(K') \cap C$ spans between contact points of C for every $C \in \mathcal{C}$ along S_i (this statement also holds for $C = T_i$ with obvious interpretation).

In particular, by property (ii), we find $C_0' \in \mathcal{C}$ such that $\varphi(C_0')$ lies along A, and $h_{C_0}(K') \cap C_0'$ spans between contact points of C_0'. Then we arrive at a contradiction just like in the previous paragraph.

Since each $C_n \in \mathcal{C}_n$ contains $\operatorname{card}(\mathcal{C}) - 1$ balls of $\mathcal{C}_{n+1}$, it is clear that Λ has no isolated points. Therefore, Λ is a Cantor set.

It remains to establish the wildness of Λ. We show that $S^3 - \Lambda$ is not simply-connected.

$D = S^3 - (\bigcup_{C \in \mathcal{C}} \operatorname{int} C \cup \text{points of contact})$ is a fundamental domain for the action of G on $S^3 - \Lambda$. Notice that the action of G can naturally be extended to an action on $\mathbf{H}^4 \cup S^3$, the compactified 4-dimensional hyperbolic space, via isometries of $\mathbf{H}^4$. The extended action has the same limit set Λ.

Consider the commutative diagram

$$\begin{array}{ccccc} & \overset{i_1}{\nearrow} & S^3 - \Lambda & \xhookrightarrow{i_2} & \mathbf{H}^4 \cup S^3 - \Lambda \\ D & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ & \underset{p}{\searrow} & S^3 - \Lambda/G & \xhookrightarrow{i_3} & \mathbf{H}^4 \cup S^3 - \Lambda/G \end{array}$$

where the vertical maps are natural projections.

Observe that D has the homotopy type of the wedge of two circles, and therefore $\pi_2 i_2 i_1 = i_3 p$ is not injective on the fundamental groups. On the other hand, if $S^3 - \Lambda$ is simply-connected, i_3 induces an isomorphism on the fundamental groups. To get a contradiction, we show that p is injective on the fundamental groups.

$S^3 - \Lambda/G$ is a 3-manifold that can be obtained from D by gluing $\partial_- C = \partial C \cap D$ with $\partial_- \varphi(C) = \partial \varphi(C) \cap D$ via h_C for $C \in \mathcal{C}$, and p is the natural quotient map. Notice that $\partial_- C \subseteq D$ is injective on the fundamental groups for every $C \in \mathcal{C}$ (a small linking circle around $A \subset E \subset S^3$ represents the commutator $[x, y]$ of the free generators x, y of $\pi_1(D)$), and therefore each identification $\partial_- C \equiv \partial_- \varphi(C)$ corresponds to an HNN extension of the fundamental group. Since a group injects into an HNN extension of itself, the result follows.

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