LINEAR SUMS OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let \( f \) belong to a certain subclass of the class of functions which are regular in the unit disc \( E = \{ z : |z| < 1 \} \). Suppose that \( \phi = \phi(f, f', f'') \) and \( \psi = \psi(f, f', f'') \) are regular in \( E \) with \( \text{Re} \phi > 0 \) in \( E \) and \( \text{Re} \psi \not> 0 \) in the whole of \( E \). In this paper we consider the following two new types of problems: (i) To find the ranges of the real numbers \( \lambda \) and \( \mu \) such that \( \text{Re}(\lambda \phi + \mu \psi) > 0 \) in \( E \). (ii) To determine the largest number \( \rho, 0 < \rho < 1 \), such that \( \text{Re}(\phi + \psi) > 0 \) in \( |z| < \rho \).

1. Introduction. Let \( A \) denote the class of functions \( f \) that are regular in the unit disc \( E = \{ z : |z| < 1 \} \) and are normalized by the conditions \( f(0) = f'(0) - 1 = 0 \). We shall denote by \( S \) the subclass of \( A \) whose members are univalent in \( E \). A function \( f \) belonging to \( S \) is said to be starlike of order \( \alpha, 0 \leq \alpha < 1 \), if \( \text{Re}\{zf'(z)/f(z)\} > \alpha, z \in E \), and we denote by \( S_t(\alpha) \) the class of all such functions. \( S_t = S_t(0) \) will be referred to as the class of starlike functions. Finally, we shall denote by \( K \) the class of convex functions, consisting of those elements \( f \in S \) which satisfy the condition \( \text{Re}(1 + z f''(z)/f'(z)) > 0 \) in \( E \). It is well known that \( K \subset S_t(1/2) \).

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are regular in \( E \), then their Hadamard product/convolution is the function denoted by \( f \ast g \) and defined by the power series
\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.
\]
It is known that \( f \ast g \) is also regular in \( E \).

Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be regular in \( E \). Then the de la Vallee Poussin mean of \( f \) of order \( n \), \( V_n(z, f) \), is defined by
\[
V_n(z, f) = \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{(n+1)(n+2) \cdots (2n)} a_n z^n.
\]

Let \( f \) be regular in \( E \) and \( g \) regular and univalent in \( E \) with \( f(0) = g(0) \). We say that \( f \) is subordinate to \( g \) in \( E \) (in symbols \( f < g \) in \( E \)) if \( f(E) \subset g(E) \).

A sequence \( \{c_n\}_{n=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) is regular and convex in \( E \), we have
\[
\sum_{n=1}^{\infty} c_n a_n z^n < f(z) \quad \text{in} \ E.
\]
In the present paper we shall mainly be concerned with the following two new types of problems:

(a) If \( \phi = (f, f', f'') \) and \( \psi = \psi(f, f', f'') \), where \( f \in K \) or \( S_t(1/2) \), such that \( \text{Re} \phi > 0 \) in \( E \) and \( \text{Re} \psi \) is not necessarily positive in the whole of the unit disc \( E \), to find the ranges of real numbers \( \lambda \) and \( \mu \) such that \( \text{Re}(\lambda \phi + \mu \psi) > 0 \) in \( E \).

(b) To find the largest number \( \rho \), \( 0 < \rho < 1 \), such that \( \text{Re}(\phi + \psi) > 0 \) in \( |z| < \rho \).

2. Preliminary results. We shall need the following results, which we state as lemmas.

**Lemma 1.** If \( f \in K \) and \( g \in S_t \), then \( (f \ast g_F)/(f \ast g) \) takes values in the convex hull of \( F(E) \) for every function \( F \) regular in \( E \).

**Lemma 2.** If \( f \) and \( g \) belong to \( S_t(1/2) \), then \( (f \ast g_F)/(f \ast g) \) takes values in the convex hull of \( F(E) \) for every function \( F \) regular in \( E \).

**Lemma 3.** A sequence \( \{c_n\}_1^\infty \) of complex numbers is a subordinating factor sequence if and only if \( \text{Re}(1 + 2 \sum_{n=1}^\infty c_n z^n) > 0 \) \(|z| < 1\).

Lemmas 1 and 2 are due to Ruscheweyh and Sheil-Small [1] and Lemma 3 is due to Wilf [2].

3. Theorems and their proofs. It is well known [1] that if \( f \in S_t(1/2) \), then \( \text{Re}(f(z)/s_n(z, f)) > 1/2 \), \( z \in E \), where \( s_n(z, f) \) denotes the \( n \)th partial sum of \( f \). From this it follows that given \( f \in S_t(1/2) \) and any two real numbers \( \lambda \geq 0 \) and \( \mu \geq 0 \), with at least one of them nonzero, then we have

\[
\text{Re} \left[ \lambda \frac{zf'(z)}{f(z)} + \mu \frac{s_n(z, f)}{f(z)} \right] > 0 \quad (z \in E).
\]

In Theorem 1 below we prove that this result continues to hold even when \( \mu \) is a suitably restricted negative or complex number.

**Theorem 1.** Let \( f \in S_t(1/2) \) and

\[
L = \text{Re} \left[ \lambda \frac{zf'(z)}{f(z)} + \mu \frac{s_n(z, f)}{f(z)} \right],
\]

where \( s_n(z, f) \) denotes the \( n \)th partial sum of \( f \). Then \( L > 0 \) in \( E \) if (i) \( \lambda \geq 0 \), \( \mu \geq 0 \) and at least one of them is nonzero, (ii) \( \mu \) is a complex number and \( \lambda > 4|\mu| \).

The result is sharp in the sense that the ranges of \( \lambda \) and \( \mu \) cannot be increased.

**Proof.** Case (i) being obvious, we take up the proof of (ii). Since \( f \) is given to be in \( S_t(1/2) \) and \( g(z) = z/(1 - z) \in K \subset S_t(1/2) \), it follows from Lemma 2 that if we choose \( F(z) = \lambda/(1 - z) + \mu(1 - z^n) \) then the function

\[
\frac{(f \ast g_F)(z)}{(f \ast g)(z)} = \frac{f(z)}{f(z)} \ast \frac{zF(z)/(1 - z)}{z/(1 - z)} = \frac{f(z) \ast [z/(1 - z)][\lambda/(1 - z) + \mu(1 - z^n)]}{f(z) \ast z/(1 - z)} = \lambda \frac{zf'(z)}{f(z)} + \mu \frac{s_n(z, f)}{f(z)},
\]

takes values in the convex hull of \( F(E) \).
Now, since by hypothesis, $\lambda > 4|\mu|$, we find that, for $z \in E$,

$$\text{Re } F(z) = \text{Re } \left[ \frac{\lambda}{1 - z} + \mu(1 - z^n) \right] \geq \frac{\lambda}{1 + r} - |\mu|(1 + r^n) > \frac{\lambda}{2} - 2|\mu| > 0$$

(equality in the second line holds at $z = -|z| = -r$ when $\mu$ is negative and $n$ is odd).

Since $(\lambda z f'(z)/f(z) + \mu s_n(z, f)/f(z))$ takes values in the convex hull of $F(E)$, assertion (ii) now follows.

To prove that the ranges of $\lambda$ and $\mu$ cannot be increased without violating the assertion of our theorem, we consider the function $f_0(z) = z/(1 - z)$ which belongs to $K$ and hence to $S_t(1/2)$. Let

$$L_0 = \text{Re } \left[ \frac{\lambda z f'_0(z)}{f'_0(z)} + \mu s_n(z, f_0) \right] = \text{Re } \left[ \frac{\lambda}{1 - z} + \mu(1 - z^n) \right] \quad (z \in E).$$

If $\lambda \geq 0$, $\mu \geq 0$ with at least one of them nonzero, then clearly $L_0 > 0$ in $E$. On the other hand if $\lambda < 0$, then $L_0 \neq 0$ in $E$ whatever $\mu$ may be. Finally, when $\lambda > 0$ and $\mu$ is negative, then it is seen that $L_0 > 0$ in $E$ only when $(\lambda/2 - 2|\mu|) > 0$. This completes the proof of our theorem.

REMARK. Since the function $f_0(z)$ also belongs to $K$, Theorem 1 remains sharp within this subclass of $S_t(1/2)$.

The significance of the following theorem emerges from the fact that if $f \in K$, then $\text{Re } (1/f'(z))$ need not be positive in the whole of the unit disc.

**THEOREM 2.** If $f \in K$, then for all $\lambda$ and $\mu$ with $\mu \geq 0$ and $\lambda > 2\mu$, we have

$$\text{Re } \left[ \frac{\lambda f(z)}{zf'(z)} + \frac{\mu}{f'(z)} \right] > 0 \quad (z \in E).$$

**PROOF.** Since $f \in K$ and the function $g(z) = z/(1 - z)^2$ is in $S_t$, in view of Lemma 1 we conclude that for all $z \in E$ the function $w$, defined by

$$w(z) = \frac{f(z) *[z/(1 - z)^2][\lambda(1 - z) + \mu(1 - z)^2]}{f(z) * z/(1 - z)^2} = \frac{\lambda f(z)}{zf'(z)} + \frac{\mu}{f'(z)},$$

takes values in the convex hull of $F(E)$, where $F(z) = \lambda(1 - z) + \mu(1 - z)^2$.

Now, since by hypothesis, $\lambda > 2\mu$, $\mu \geq 0$, we find that

$$\text{Re } F(z) = \text{Re } [\lambda(1 - z) + \mu(1 - z)^2] = (\lambda - 2\mu)(1 - r \cos \theta) + \mu[(1 - r^2) + 2(1 - r \cos \theta)^2] \quad (z = re^{i\theta}) > 0, \quad z \in E.$$

The assertion of Theorem 2 is now clear.

The fact that for every $f \in S_t(1/2)$, $\text{Re } f'(z) > 0$ only in $|z| < 1/\sqrt{2} = 0.707 \ldots$ underlines the importance of our next theorem.
THEOREM 3. If \( f \in S_t(1/2) \), then
\[
\text{Re} \left( \frac{f(z)}{z} + f'(z) \right) > 0
\]
in \( |z| < \rho = \sqrt{4\sqrt{2} - 5} \approx 0.81 \ldots. \) The number \( \rho \) is the best possible one.

PROOF. Consider the function
\[
h(z) = \frac{1}{1 - z} + \frac{1}{(1 - z)^2} \quad (z \in E).
\]

We first proceed to prove that \( \text{Re} h(z) > 0 \) in \( |z| < \rho = \sqrt{4\sqrt{2} - 5} \approx 0.81 \ldots. \)

Letting \( \frac{1}{1 - z} = \text{Re} e^{i\phi} \), we get
\[
\frac{1}{1 + r} < R < \frac{1}{1 - r} \quad (|z| = r)
\]
and
\[
\cos \phi = \frac{1 + R^2 - r^2R^2}{2R} \quad (\leq 1).
\]

(1) and (3) provide
\[
2 \text{Re} h(z) = 2|R \cos \phi + R^2 \cos 2\phi|
= 2 + (1 - 3r^2)t + (1 - r^2)^2t^2 \quad (t = R^2)
= \psi(t), \text{ say.}
\]

It is now readily verified that for \( r \geq \sqrt{7} - 2 \), \( t_1 \) given by \( t_1 = (3r^2 - 1)/(2(1 - r^2)^2) \) lies in the range of \( t = R^2 \) and that \( \partial \psi / \partial t = 0 \) and \( \partial^2 \psi / \partial t^2 > 0 \) at \( t = t_1 \). We, therefore, conclude that for \( r \geq \sqrt{7} - 2 \),
\[
\min \psi(t) = \psi(t_1) = \frac{8(1 - r^2)^2 - (3r^2 - 1)^2}{4(1 - r^2)^2} > 0,
\]
if \( r < \rho = \sqrt{4\sqrt{2} - 5} \approx 0.81 \ldots. \)

On the other hand, if \( r < \sqrt{7} - 2 \), then one can easily see that
\[
\min \psi(t) = \psi \left( \frac{1}{(1 + r)^2} \right) = \frac{2(2 + r)}{(1 + r)^2} > 0.
\]

To sum up, we have shown that
\[
\text{Re} h(z) > 0 \quad \text{in} \quad |z| < \rho = \sqrt{4\sqrt{2} - 5},
\]
from which it follows that
\[
\text{Re} h(\rho z) > 0, \quad z \in E.
\]

Now taking \( g(z) = z \) and \( F(z) = \rho h(\rho z) \) in Lemma 2, we conclude that the values of the function
\[
g(z) = \frac{f(z) \ast z[\rho h(\rho z)]}{f(z) \ast z}
= \frac{f(z) \ast z[\rho/(1 - \rho z) + \rho/(1 - \rho z)^2]}{f(z) \ast z}
= \rho \left[ \frac{f(\rho z)}{\rho z} + f'(\rho z) \right]
\]
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lie in the convex hull of \( F(E) \). However, in view of (4) we have

\[
\text{Re} \ F(z) = \text{Re} \ \rho h(\rho z) > 0 \quad \text{in } E.
\]

We have thus proved that

\[
\text{Re} \left[ \frac{f(\rho z)}{\rho z} + f'(\rho z) \right] > 0
\]

in \( E \) and hence

\[
\text{Re} \left[ \frac{f(z)}{z} + f'(z) \right] > 0 \quad \text{in } |z| < \rho.
\]

If we consider the function \( f_0(z) = z/(1 - z) \in K \subset S_t(1/2) \) then it is seen that

\[
\frac{f_0(z)}{z} + f'_0(z) = \frac{1}{1 - z} + \frac{1}{(1 - z)^2},
\]

and the assertion regarding the sharpness of the number \( \rho \) now becomes obvious in view of the definition of the function \( h \).

**Theorem 4.** If \( f \in K \), then

\[
\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \frac{1}{f'(z)} \right] > 0
\]

in \( |z| < \rho = (\sqrt{5} - 1)/\sqrt{2} = 0.874 \ldots \). The number \( \rho \) cannot be replaced by any larger one.

**Proof.** Proceeding as in the proof of the previous theorem, one can show that the function \( h \), defined by

\[
h(z) = \frac{2}{1 - z} - 1 + (1 - z)^2 \quad (z \in E),
\]

has the property that \( \text{Re} \ h(z) > 0 \) only when \( |z| < \rho = (\sqrt{5} - 1)/\sqrt{2} = 0.874 \ldots \), from which it follows that

\[
\text{Re} \ h(\rho z) > 0 \quad \text{in } E.
\]

Since the function \( f \) is given to be in \( K \) and \( g(z) = z/(1 - \rho z)^2 \) belongs to \( S_t \), Lemma 1, in conjunction with (6), provides that the function \( p \), defined by

\[
p(z) = \frac{f(z) * g(z) h(\rho z)}{f(z) * g(z)}
\]

has positive real part in \( E \). The desired conclusion is now obvious.

The sharpness of the number \( \rho \) follows from the fact that for the function \( f_0(z) = z/(1 - z) \in K \) we have

\[
\left( 1 + \frac{zf''_0(z)}{f'_0(z)} \right) + \frac{1}{f'_0(z)} = \frac{2}{1 - z} - 1 + (1 - z)^2
\]

\[
= h(z) \quad \text{(given by (5))},
\]

and that \( \text{Re} \ h(z) > 0 \) only when \( |z| < \rho = (\sqrt{5} - 1)/\sqrt{2} \).

We observe that the disc \( |z| < \rho = 0.874 \ldots \) is much larger than the disc \( |z| < \sqrt{2}/2 = 0.707 \) in which \( \text{Re}(1/f'(z)) > 0 \) for every \( f \in K \).

We omit the proof of the following theorem.
THEOREM 5. If \( f \in S_t(1/2) \), then
\[
\text{Re} \left[ \frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right] > 0
\]
in \(|z| < \rho = \sqrt{8\sqrt{2} - 11} = 0.56 \ldots \). The number \( \rho \) is the best possible one.

If \( f \in K \), then it is well known that \( g(z) = (f(z) - f(-z))/2 \) is an odd function in \( S_t \) and hence \( \text{Re}(z/g(z)) > 0 \) in \( E \). Our next theorem generalizes this latter result.

THEOREM 6. Let \( f \in K \). Then for each integer \( n \geq 1 \) we have
\[
\text{Re} \left( \frac{v_n(z, f) - v_n(-z, f)}{f(z) - f(-z)} \right) > 0 \quad (z \in E),
\]
where \( v_n(z, f) \) is the de la Vallée Poussin mean of \( f \) of order \( n \).

PROOF. Let us first suppose that \( n \) is an odd integer, \( n = 2m + 1 \), say, and consider the function \( F_{2m+1} \) defined by
\[
F_{2m+1}(z) = 2(1 - z^2) \left[ \frac{2m+1}{2m+2} + \frac{(2m+1)2m(2m-1)}{(2m+2)(2m+3)(2m+4)} z^2 
+ \frac{(2m+1)(2m)(2m-1)(2m-2)(2m-3)}{(2m+2)(2m+3)(2m+4)(2m+5)(2m+6)} z^4 
+ \cdots + \frac{(2m+1)(2m) \cdots 1}{(2m+2)(2m+3) \cdots (2(2m+1))} z^{2m} \right].
\]

Obviously \( F_{2m+1} \) is regular in \( E \) (in fact it is an entire function), and we can write it in the form
\[
F_{2m+1}(z) = 2 \left[ \frac{2m+1}{2m+2} - \frac{2m+1}{2m+2} \left\{ 1 - \frac{2m(2m-1)}{(2m+3)(2m+4)} \right\} z^2 
- \frac{(2m+1)2m(2m-1)}{(2m+2)(2m+3)(2m+4)} \left\{ 1 - \frac{(2m-2)(2m-3)}{(2m+5)(2m+6)} \right\} z^4 
- \cdots - \frac{(2m+1)2m(2m-1) \cdots 3}{(2m+2)(2m+3) \cdots (2m+(2m-1))(4m)} \right] z^{2m+2}.
\]

In view of (8) and (7) it is now easy to see that in \( E \) we have
\[
\text{Re} F_{2m+1}(z) > 0.
\]

Next suppose that \( n \) is an even integer, \( n = 2m \), say, and consider the function \( F_{2m} \) defined by
\[
F_{2m}(z) = 2(1 - z^2) \left[ \frac{2m}{2m+1} + \frac{2m(2m-1)(2m-2)}{(2m+1)(2m+2)(2m+3)} z^2 
+ \cdots + \frac{2m(2m-1) \cdots 3 \cdot 2z^{2m-2}}{(2m+1)(2m+2) \cdots (2m+(2m-1))} \right].
\]

As before, one can see that \( \text{Re} F_{2m}(z) > 0 \) in \( E \).
In Lemma 1, letting \( g(z) = z/(1 - z^2) \), a function belonging to \( S_t \), and

\[
F(z) = \begin{cases} 
F_{2m+1}(z) & \text{if } n = 2m + 1 \text{ is odd,} \\
F_{2m}(z) & \text{if } n = 2m \text{ is even,}
\end{cases}
\]

we conclude that for every integer \( n \geq 1 \), the function

\[
w(z) = \frac{f(z) * zF(z)/(1 - z^2)}{f(z) * z/(1 - z^2)}
\]

takes values in the right half-plane, that is, \( \text{Re} \, w(z) > 0 \) in \( E \).

A moderate calculation, however, shows that

\[
w(z) = \frac{v_n(z, f) - v_n(-z, f)}{f(z) - f(-z)}.
\]

This completes the proof of our theorem.

As observed earlier, if \( f \in K \), then \( g(z) = (f(z) - f(-z))/2 \) is an odd starlike function. We conclude this paper with a theorem pertaining to \( g \) which, although not in tune with the earlier one, is of considerable interest.

**Theorem 7.** If \( f \in K \), then

\[ g(\rho z) < f(z) \]

in \( E \), where

\[ g(z) = \frac{1}{2} (f(z) - f(-z)) \]

and \( \rho = \sqrt{2} - 1 = 0.414 \ldots \) The number \( \rho \) cannot be replaced by any larger one.

**Proof.** Since \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K \) and

\[ g(\rho z) = z + a_3 \rho^3 z^3 + a_5 \rho^5 z^5 + \cdots , \]

the conclusion of our theorem would follow provided \( \{\rho, 0, \rho^3, 0, \rho^5, \ldots \} \) is a subordinating factor sequence. In view of Lemma 3 this will be true if and only if

\[
\text{Re} \left[ 1 + 2 \sum_{m=0}^{\infty} \rho^{2m+1} z^{2m+1} \right] = \text{Re} \left[ 1 + \frac{2\rho z}{1 - \rho^2 z^2} \right] > 0 \quad (z \in E),
\]

or,

\[ 1 - 2\rho/(1 - \rho^2) \geq 0, \]

which is true by the choice of \( \rho \).

To prove that the number \( \rho \) is the best possible one, let us consider the function \( f(z) = z/(1 - z) \in K \). It is then seen that for any \( 0 \leq \lambda \leq 1 \), \( g(\lambda z) = \lambda z/(1 - \lambda^2 z^2) \). Since \( g(-\lambda) = -\lambda/(1 - \lambda^2) < -1/2 \) if \( \lambda > \sqrt{2} - 1 \), from the fact that the range of \( f \) is the half-plane \( \{w | \text{Re} \, w > -1/2\} \) it follows that \( g(\lambda z) \) cannot be subordinate to \( f \) in \( E \) if \( \lambda > \sqrt{2} - 1 = \rho \).

**References**