CHARACTERIZING $S^m$ BY THE SPECTRUM OF THE LAPLACIAN ON 2-FORMS

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ABSTRACT. The Euclidean sphere $S^{2n+1}$ is characterized by the spectrum of the Laplacian on 2-forms in all dimensions.

1. Introduction. It was recently shown [4, 5] that within the class of Kaehler manifolds, complex projective $n$-space $CP_n$ with the Fubini metric $g_0$ is characterized by the spectrum of the Laplacian on 2-forms in all dimensions. More precisely, let $(M, g)$ be a compact Kaehler manifold with $Spec^2(M, g) = Spec^2(CP_n, g_0)$, where $Spec^p(M, g)$ denotes the spectrum of the Laplacian with respect to the Kaehler metric $g$ on $p$-forms of $M$. Then, $(M, g)$ is holomorphically isometric to $(CP_n, g_0)$ for all $n$. In this paper, we consider the problem of characterizing the constant curvature sphere $S^m$ by the spectrum of its Laplacian on $p$-forms: If $Spec^p(M, g) = Spec^p(S^m, g_0)$ for some fixed $p$, is $(M, g)$ isometric with $(S^m, g_0)$, where $g_0$ is the constant curvature metric, that is, does there exist a diffeomorphism $f: M \to S^m$ such that $f^*g_0 = g$? The answer to this question is yes in the following cases:

(a) $p = 0$ and $m \leq 6$ [1, 8];
(b) $p = 1$ and $m = 2, 3, 16, \ldots, 93$ [9];
(c) $p = 2$ and $m = 2, 3, 6, 7, 14, 17, \ldots, 178$ [11].

REMARK. Patodi [7] proved that if $Spec^p(M, g) = Spec^p(S^m, g_0)$ for $p = 0$ and 1, then $(M, g)$ is isometric with $(S^m, g_0)$ in all dimensions.

In order to obtain uniqueness for a fixed $p$ in all (odd) dimensions we confine ourselves to the class of normal contact Riemannian manifolds and obtain the following statement.

THEOREM 1. Let $(M, g)$ be a compact normal contact Riemannian manifold. If $Spec^2(M, g) = Spec^2(S^{2n+1}, g_0)$, where $g_0$ is the metric of constant curvature $k = 1$, then $g$ is a metric of the same constant curvature $k = 1$.

2. The spectrum. Let $(M, g)$ be a compact connected $C^\infty$ Riemannian manifold without boundary, and with Laplacian $\Delta = -(dd^* + d^*d)$, where $d$ is the operator of exterior differentiation and $d^*$ is its adjoint with respect to the Riemannian metric $g$. Then, for each $p = 0, 1, 2, \ldots$, the spectrum of $\Delta$ is given by

$$Spec^p(M, g) = \{0 \geq \lambda_{1,p} \geq \lambda_{2,p} \geq \cdots \geq \lambda_{k,p} \geq \cdots \downarrow -\infty\}.$$
each eigenvalue $\lambda_{i,p}$ repeated as often as its multiplicity. By Hodge theory, $0 \in \text{Spec}^p(M, g)$ if and only if the $p$th betti number $b_p(M)$ is not zero, and its multiplicity is then $b_p(M)$. For $p = 2$, the Minakshisundaram-Pleijel-Gaffney formula is

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,2}t) = \frac{1}{(4\pi t)^m} \sum_{i=0}^{N} a_{i,2} t^i + O(t^{N-m+1}), \quad t \downarrow 0,$$

the coefficients $a_{i,2}$, $i = 0, 1, 2$, being given by

$$a_{0,2} = \frac{m(m-1)}{2} V, \quad V = \text{vol}(M),$$

$$a_{1,2} = \frac{m^2 - 13m + 24}{2} \int_M \rho \, dV,$$

$$a_{2,2} = \frac{1}{720} \int_M \left[ 2(m^2 - 31m + 240) |R|^2 - 2(m^2 - 181m + 1080) |S|^2 
+ 5(m^2 - 25m + 120) \rho^2 \right] \, dV,$$

where $R$, $S$, and $\rho$ denote the curvature tensor, the Ricci tensor, and the scalar curvature of $g$, respectively, and $|R|^2 = \sum R_{ijkl} R^{ijkl}$, $|S|^2 = \sum R_{ij} R^{ij}$, $R_{ijkl}$ and $R_{ij}$ denoting the components of $R$ and $S$, respectively [7].

If $\text{Spec}^2(M, g) = \text{Spec}^2(M', g')$, then $\dim M = \dim M'$, $V = V'$, $b_2(M) = b_2(M')$, and

$$\int_M \rho \, dV = \int_{M'} \rho' \, dV' \quad \text{and} \quad a_{2,2} = a'_{2,2}.$$

The following expression for $a_{2,2}$ will be useful:

$$720 a_{2,2} = \int_M \left[ Q_1 |C|^2 + Q_2 \left( |S|^2 - \frac{\rho^2}{m} \right) + Q_3 \rho^2 \right] \, dV,$$

where $|C|^2 = \sum C_{ijkl} C^{ijkl}$ is the square of the norm of the Weyl conformal curvature tensor. The components of $C$ are

$$C_{ijkl} = R_{ijkl} - \frac{1}{m-2} \left( R_{ijk} g_{il} - R_{ijl} g_{ik} + g_{jk} R_{il} - g_{jl} R_{ik} \right)$$

$$+ \frac{\rho}{(m-1)(m-2)} \left( g_{jk} g_{il} - g_{jl} g_{ik} \right),$$

and

$$Q_1 = \frac{2(m-15)(m-16)}{m-2},$$

$$Q_2 = \frac{8(m-15)(m-16)}{m-2} - 2(m^2 - 181m + 1080),$$

$$Q_3 = \frac{4(m-15)(m-16)}{m(m-1)} - \frac{2(m^2 - 181m + 1080)}{m} + 5(m^2 - 25m + 120).$$

Note that $|S|^2 \geq \rho^2/m$ with equality holding if and only if $g$ is an Einstein metric.
If \((M', g')\) is a manifold of constant curvature \(k'\), then \(|C'| = 0\) and \(|S'|^2 = \rho^2/m\), so by (2.5)

\[
(2.6) \quad \int_M \left[ Q_1 |C|^2 + Q_2 \left( |S|^2 - \frac{\rho^2}{m} \right) + Q_3 \rho^2 \right] dV = \int_{M'} Q_3 \rho'^2 dV'.
\]

Thus, since \(Q_1, Q_2,\) and \(Q_3\) are positive for \(m = 3, 6, 7, 14, 17, 18, \ldots, 178\), and

\[
\int_M \rho^2 dV \geq \int \rho'^2 dV',
\]

the latter being a consequence of Schwarz's inequality and (2.4), \(g\) is a conformally flat Einstein metric. Hence, \((M, g)\) is a manifold of constant curvature \(k = k'\) in these dimensions [11]. For \(m = 8\), \(Q_3\) vanishes and \(Q_1\) and \(Q_2\) are both positive, so again \(g\) is a constant curvature metric. For \(m = 15\) and 16, \(Q_1\) vanishes and \(Q_2\) and \(Q_3\) are both positive, so \(g\) is an Einstein metric with scalar curvature \(\rho'\). If \(M' = S^m\) with the metric of constant curvature \(k'\), then, since \(V = V'\) it follows from [2, p. 257] that \((M, g)\) is isometric with \((S^m, g')\) for \(m = 15\) and 16. This extends Theorem 3.1 in [11].

The case \(\rho = \text{constant}\) is interesting. For, since \(Q_1\) and \(Q_2\) are positive for \(m = 9, \ldots, 13\), we may again conclude that \(g\) is a metric of constant curvature.

**Theorem 2.** Let \((M, g)\) be a compact Riemannian manifold. If \(\text{Spec}^2(M, g) = \text{Spec}^2(S^m, g_0)\), where \(g_0\) is a metric of constant curvature \(k'\), then \(g\) is a metric of constant curvature \(k = k'\) for \(m = 2, 3, 6, 7, 8, 14, 15, 16, 17, \ldots, 178\). If, in addition, \(g\) is a metric of constant scalar curvature, then \(g\) is a metric of constant curvature \(k = k'\) for \(m = 2, 3, 6, \ldots, 178\).

The case \(m = 2\) is a consequence of the fact that \(\text{Spec}^2(M, g) = \text{Spec}^2(S^2, g_0)\) implies \(\text{Spec}^0(M, g) = \text{Spec}^0(S^2, g_0)\).

**Theorem 3.** Let \((M, g)\) be a compact Riemannian manifold with \(\text{Spec}^2(M, g) = \text{Spec}^2(S^m, g_0)\), where \(g_0\) is a metric of constant curvature \(k'\), and for some \(\lambda \in \mathbb{R}\), let

\[
(2.7) \quad \int_M (|S|^2 - \lambda \rho^2) dV = \int_{S^m} (|S'|^2 - \lambda \rho'^2) dV',
\]

where the prime indicates corresponding quantities in \((S^m, g_0)\). Then, if

(i) \(\lambda < 1/m\), \(g\) is a metric of constant curvature \(k = k'\),

(ii) \(\lambda \geq 1/m\), \(g\) is a metric of constant curvature \(k = k'\) for each \(m\) satisfying \((\lambda - 1/m)Q_2 + Q_3 > 0\).

**Proof.** Since for \(m = 15, 16\) the theorem follows from Theorem 2, we will assume that \(m \neq 15, 16\) and therefore \(Q_1 > 0\). Formula (2.6) may be rewritten in the form

\[
(2.8) \quad \int_M \left[ Q_1 |C|^2 + \mu Q_2 \left( |S|^2 - \frac{\rho^2}{m} \right) + (1 - \mu) Q_2 \left( (|S|^2 - \lambda \rho^2) + \left( \lambda - \frac{1}{m} \right) \rho^2 \right) + Q_3 (\rho^2 - \rho'^2) \right] dV = 0,
\]
where $\mu$ is a real number to be specified later. By (2.7) and the fact that $|S'|^2 = \rho'^2/m$, formula (2.8) becomes

$$\int_M \left[ Q_1 |C|^2 + \mu Q_2 \left( |S|^2 - \frac{\rho^2}{m} \right) \\
+ \left( \lambda - \frac{1}{m} \right) Q_2 + Q_3 - \left( \lambda - \frac{1}{m} \right) \mu Q_2 \right] (\rho^2 - \rho'^2) \, dV = 0. \tag{2.9}$$

If $\lambda < 1/m$, we specify $\mu$ as follows: Take $\mu$ to be of the same sign as $Q_2$ (note that $Q_2 \neq 0$ for any integer $m$), and $|\mu|$ to be so large that $(\lambda - 1/m)Q_2 + Q_3 - (\lambda - 1/m)\mu Q_2 > 0$. Since $|S|^2 - \rho^2/m \geq 0$ and $\int (\rho^2 - \rho'^2) \, dV \geq 0$, (2.9) implies that $C = 0$, $\rho = \rho'$, and $|S|^2 = \rho^2/m$. The last equality shows that $g$ is Einstein, and so $g$ is a metric of positive constant curvature.

If $\lambda \geq 1/m$, we take $\mu = 0$. Formula (2.9) then becomes

$$\int_M \left[ Q_1 |C|^2 + \left( \lambda - \frac{1}{m} \right) Q_2 + Q_3 \right] (\rho^2 - \rho'^2) \, dV = 0.$$

Since $m$ satisfies the inequality $(\lambda - 1/m)Q_2 + Q_3 > 0$, $C = 0$ and $\rho = \rho'$. Hence, by (2.7)

$$\int_M \left( |S|^2 - \frac{\rho^2}{m} \right) \, dV = \int_M \left( |S'|^2 - \lambda \rho^2 \right) + \left( \lambda - \frac{1}{m} \right) \rho^2 \right) \, dV'$$

$$= \int_{S'} \left( |S'|^2 - \lambda \rho'^2 \right) + \left( \lambda - \frac{1}{m} \right) \rho'^2 \right) \, dV' = 0.$$

But $|S|^2 \geq \rho^2/m$, so $|S|^2 = \rho^2/m$, that is, $g$ is an Einstein metric. Since $C = 0$, $g$ is a constant curvature metric.

3. Contact manifolds. An $m(= 2n + 1)$-dimensional $C^\infty$ manifold is called a contact manifold if it carries a global 1-form $\eta$, called the contact form, with the property $\eta \wedge (d\eta)^n \neq 0$ everywhere. The classical example is the bundle of unit tangent vectors to an oriented $(n + 1)$-dimensional manifold. An odd-dimensional sphere $S^{2n+1}$ possesses a contact structure which is not of this type. More generally, a smooth hypersurface of $(2n+2)$-dimensional affine space with the property that no tangent space contains the origin has a contact structure. J. Martinet showed that every compact 3-manifold carries a contact structure. A compact Hodge manifold $B$ has a contact manifold canonically associated with it as a circle bundle with $B$ as base space. Thus, the class of contact manifolds is quite extensive.

An almost contact structure $(\phi, X_0, \eta)$ on a $(2n + 1)$-dimensional $C^\infty$ manifold $M$ is given by a linear transformation field $\phi$, a vector field $X_0$, and a 1-form $\eta$ satisfying

$$\eta(X_0) = 1, \quad \phi X_0 = 0, \quad \text{and} \quad \phi^2 = -I + \eta \otimes X_0. \tag{3.1}$$

In this case, a Riemannian metric $g$ can be found such that

$$\eta = g(X_0, \cdot) \quad \text{and} \quad g(\phi X, Y) = -g(X, \phi Y) \tag{3.2}$$

for any vector fields $X$ and $Y$. 

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A contact manifold with contact form $\eta$ has an underlying almost contact Riemannian structure $(\phi, X_0, \eta, g)$ such that $g(X, \phi Y) = d\eta(X, Y)$. If the almost complex structure $J$ on $M \times R$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - fX_0, \eta(X)\frac{d}{dt})$$

is integrable, the almost contact structure is said to be normal, and $M$ is said to be a normal contact manifold. In this case, the unit vector field $X_0$ is a Killing vector field. Moreover,

$$g(R(X, X_0)Y, X_0) = g(X, Y) - \nu(X)\nu(Y) = g(\phi X, \phi Y)$$

and

$$S(X, X_0) = 2n\eta(X).$$

The sectional curvature of a plane section containing $X_0$ is therefore positive, and the Ricci curvature in the direction $X_0$ is $2n$.

The standard contact Riemannian structure on an odd-dimensional sphere is normal.

Put $S(X, Y) = S(X, \phi Y)$. Then $S$ is a skew symmetric bilinear form on $M$.

**Lemma 1.** Let $M^{2n+1}$ be a compact normal contact Riemannian manifold with $b_2(M) = 0$. Then there exists a 1-form $\alpha$ on $M$ such that $S = d\alpha$ and $\alpha(X_0) = \text{const.}$

**Proof.** It is shown in [6] that $S$ is closed, so since $b_2(M) = 0$, $S$ is exact. We may therefore write $S = d\beta$. Let $H$ denote the group of isometries of $(M, g)$ preserving $S$. Then $H$ is a compact Lie group. Let $H_0$ be the 1-parameter group of diffeomorphisms of $M$ generated by $X_0$. Then, $H_0$ is a subgroup of $H$. Indeed, since $X_0$ is a Killing vector field, the elements of $H_0$ are isometries. In addition since $\tilde{S}(X_0, \cdot) = 0$, we obtain $L_{X_0}\tilde{S} = (i(X_0)d + di(X_0))\tilde{S} = 0$. Therefore, the elements of $H_0$ preserve $\tilde{S}$.

Set $\alpha = \int_H h^*(\beta) \, dh$, where $h$ is an arbitrary element of $H$ and $dh$ is the invariant measure on $H$ normalized by the condition $\int_H dh = 1$. Then,

$$d\alpha = \int_H h^*(d\beta) \, dh = \int_H h^*(\tilde{S}) \, dh = \int_H \tilde{S} \, dh = \tilde{S}.$$  

Clearly, $h^*(\alpha) = \alpha$ for any $h \in H$. Since $H_0 \subset H$ it follows that $L_{X_0}\alpha = 0$. Therefore, since $di(X_0)\alpha = L_{X_0}\alpha - i(X_0)d\alpha = 0$, we conclude that $i(X_0)\alpha = \text{const.}$

**Proof of Theorem 1.** Let $(M, g)$ be a compact normal contact Riemannian manifold, and let $(\phi, X_0, \eta, g)$ be its underlying almost contact Riemannian structure. Set $\Phi(X, Y) = g(X, \phi Y)$. Then, $\Phi$ is skew symmetric. The following formulas are known (see [3]):

$$\nabla_X X_0 = -\phi X,$$

$$\langle \nabla_X \eta \rangle(Y) = \Phi(X, Y),$$

$$\langle \nabla_X \phi \rangle Y = g(X, Y)X_0 - \eta(Y)X,$$

$$\langle \nabla_X \Phi \rangle(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$$

for any vector fields $X, Y, Z$. 

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Using (3.4)-(3.8) we obtain the following formulas by direct computation:

\[ d^* \Phi = 2n\eta, \]

\[ d^* \Phi^2 = 4(n - 1)\eta \land \Phi, \]

\[ i(\Phi)\tilde{S} = \frac{1}{2}(\rho - 2n), \]

\[ i(\Phi^2)\tilde{S}^2 = \frac{1}{2}(\rho^2 - 2|S|^2 - 4n\rho + 12n^2), \]

\[ i(\tilde{S})(\eta \land \Phi) = \frac{1}{2}(\rho - 2n)\eta, \]

where \( i \) is the adjoint of exterior multiplication, that is, if \( (\ , \) \) denotes the local scalar product with respect to the Riemannian metric \( g, \)

\[ (i(\alpha)\beta, \gamma) = (\beta, \alpha \land \gamma), \]

where \( \alpha, \beta, \) and \( \gamma \) are forms of degrees \( p, q, \) and \( q - p, \) respectively. Denote by \( (\alpha, \alpha') = \int_M (\alpha, \alpha') \, dV \) the global scalar product. By (3.9), (3.11), and Lemma 1

\[ \frac{1}{2} \int_M (\rho - 2n) \, dV = (i(\Phi)\tilde{S}, 1) = (\tilde{S}, \Phi) = (d\alpha, \Phi) = (\alpha, d^* \Phi) = 2n(\alpha, \eta) = 2n\alpha(X_0)V. \]

On the other hand, by (2.4)

\[ \frac{1}{2} \int_M (\rho - 2n) \, dV = \frac{1}{2} \int_{S^m} (\rho' - 2n) \, dV' = 2n^2V \]

since \( \rho' = 2n(2n + 1). \) Therefore, \( \alpha(X_0) = n. \) Now, by (3.10), (3.12) and (3.13)

\[ \frac{1}{2} \int_M (\rho^2 - 2|S|^2 - 4n\rho + 12n^2) \, dV \\
= (i(\Phi^2)\tilde{S}^2, 1) = (\tilde{S}^2, \Phi^2) \\
= (d(\alpha \land d\alpha), \Phi^2) = (\alpha \land \tilde{S}, d^*\Phi^2) \\
= 4(n - 1)(\alpha \land \tilde{S}, \eta \land \Phi) = 4(n - 1)(\alpha, i(\tilde{S})(\eta \land \Phi)) \\
= 2(n - 1)\alpha(X_0) \int_M (\rho - 2n) \, dV \\
= 8n^3(n - 1)V. \]

Therefore, \( \int_M (\rho^2 - 2|S|^2) \, dV = 16n^3(n - 1)V + \int_{S^m} (4n\rho - 12n^2) \, dV = (16n^4 - 4n^2)V. \)

On the other hand, for \( (S^m, g_0), \)

\[ \int_{S^m} (\rho^2 - 2|S'|^2) \, dV' = (16n^4 - 4n^2)V, \]

so we obtain

\[ \int_M (|S|^2 - \frac{\rho^2}{2}) \, dV = \int_{S^m} (|S'|^2 - \frac{\rho'^2}{2}) \, dV'. \]

Applying Theorem 3 with \( \lambda = \frac{1}{2} \) and noting that \( (\frac{1}{2} - \frac{1}{m})Q_2 + Q_3 > 0 \) for all odd \( m \geq 3, \) we conclude that \( g \) is a metric of constant curvature \( k = 1 \) for all \( n \geq 1. \)

We thank the referee for mentioning the following result due to Tsagas [10]: For a given dimension \( m \) there exists an integer \( p \in [0, m] \) such that if \( \text{Spec}^p(M, g) = \text{Spec}^p(S^m, g_0), \) then \( g \) is a metric of the same constant curvature as \( g_0. \)
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