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SHORTER NOTES

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AN ELEMENTARY PROOF OF SŁODKOWSKI'S THEOREM

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ABSTRACT. Let $\mathfrak{A}$ be a complex Banach algebra, and $\sigma(x)$ be the spectrum of $x \in \mathfrak{A}$. We give a very short proof that if $f\colon G \to \mathfrak{A}$ is holomorphic (G open in $\mathbf{C}$), then $\sigma \circ f\colon G \to 2^{\mathbf{C}}$ is Oka-analytic.

The following result appeared in [**4**, Corollary 3.3, p. 371] in the case $\mathfrak{A} = \mathcal{B}(X)$, X being a complex Banach space.

THEOREM. *If G is open in $\mathbf{C}$, $\mathfrak{A}$ is a complex Banach algebra* (*with identity element e*), *and if $\lambda \mapsto a_\lambda\colon G \to \mathfrak{A}$ is a vector-valued analytic function, then $\lambda \mapsto \sigma(a_\lambda)\colon G \to 2^{\mathbf{C}}$ is an Oka-analytic multivalued function* (*viz.*, $U = \{(\lambda, z) \in G \times \mathbf{C}\colon z \notin \sigma(a_\lambda)\}$ *is a holomorphically convex open set in* $\mathbf{C}^2$).

The same author, and various others, have reexamined and developed it several times, giving also alternate proofs ([**1**, Theorem 3.2, p. 46; **3**, Theorem 5.1, p. 56], and the literature cited there); an elementary one is presented here, based on the trivial extension of the following classical elementary lemma of complex analysis to functions taking values in a Banach space.

LEMMA ("Simultaneous Extendability"; see e.g. [2, Lemma, p. 161]). *If the open set $U \subset \mathbf{C}^n$ is not holomorphically convex, then there exists a polydisk $B \not\subseteq U$ centered at $u \in U$ such that the restriction to the connected component V of $B \cap U$ containing u of every holomorphic function on U extends holomorphically to B.*

PROOF OF THE THEOREM. Since $\mathfrak{A}^{-1} = \{\text{invertible elements of } \mathfrak{A}\}$ is open in $\mathfrak{A}$, and $(\lambda, z) \mapsto f(\lambda, z) = a_\lambda - ze\colon G \times \mathbf{C} \to \mathfrak{A}$ is continuous, then $U = f^{-1}(\mathfrak{A}^{-1})$ is open in $\mathbf{C}^2$. Assume U is not holomorphically convex: with the notations of the lemma, let $u_0 = (\lambda_0, z_0) \in B \cap \partial V$ (so $u_0 \in \partial U$). If $\lambda_0 \in \partial G$, then the restriction to V of $(\lambda, z) \mapsto (\lambda - \lambda_0)^{-1}\colon U \to \mathbf{C}$ could not extend to u_0. Else if $\lambda_0 \in G$, then the restriction of $(\lambda, z) \mapsto R(\lambda, z) = (a_\lambda - ze)^{-1}\colon U \to \mathfrak{A}$ to V extends to u_0, and relations $R(\lambda, z) \cdot (a_\lambda - ze) = e = (a_\lambda - ze) \cdot R(\lambda, z)$ also would extend to it, thus yielding the invertibility of $a_{\lambda_0} - z_0 e$: but $z_0 \in \sigma(a_{\lambda_0})$. In both cases, a contradiction. □

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