PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

A NOTE ON PUSHING UP

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ABSTRACT. The amalgam method is used to handle some so-called pushing-up problems in finite groups.

1. Introduction. Generally speaking, pushing-up problems in finite group theory have been posed in terms of obstructions:

Suppose $S \in \mathrm{Syl}_p(P)$ such that no characteristic subgroup of S is normal in P. What then can we say about S? In Goldschmidt's fundamental paper [**G**], he introduced what is now called the amalgam method. This was later in particular applied to a pushing-up problem by Stellmacher [**S**] that simplified the works of Niles [**N**] and Baumann [**B**]. So let us reformulate our pushing-up question in terms of amalgams. (See [**G**] for the basic definitions.)

Suppose P and H are abstract finite groups. We say $P \supseteq S \subseteq H$ is a *pushing-up amalgam* for $P/O_p(P)$ if

(1) $S \in \mathrm{Syl}_p(P)$.

(2) No nontrivial subgroup of S is normal in P and H.

(3) $S \trianglelefteq H$.

Take X to be a finite group and $S \in \mathrm{Syl}_p(X)$. Notice that if M is a maximal p-local but not a p-parabolic of X (i.e., $N_X(S) \nsubseteq M$) then one clearly has a pushing-up amalgam. Or if $N_X(S)$ is contained in a unique maximal p-local M of X, then one also has a pushing-up amalgam, unless of course M is strongly embedded. We further remark that there is some motivation in changing (1) to require simply that S be a p-group containing $O_p(P)$. (See [**Gn**, p. 280] and [**C**].) This actually does not entail any essential changes in the argument of our theorem below.

So as a general goal, given any pushing-up amalgam for a chev(p)-group one would like to be able to describe S. In this note we give a small indication to be optimistic about solving this problem in general. (See also [**M**].) Specifically, we consider pushing-up amalgams for chev(p)-groups satisfying Hypothesis A below. First, we define a $\mathrm{GF}(p)X$-module W to be an FF-module if $C_X(W)$ is a p'-group and X has a nontrivial elementary abelian p-group A such that $|W : C_W(A)| \leq |A|$. A is called an offending subgroup.

HYPOTHESIS A. *X is a group where $X/\phi(X)$ is simple. For any irreducible* FF, $\mathrm{GF}(p)X$*-module W and any quadratic offending subgroup A of X, we have*

(1) $|W : C_W(A)| = |A|$.

(2) *A is a maximal quadratic subgroup of X.*

(3) *If $A \subseteq S \in \mathrm{Syl}_p(X)$, then there exist $g \in X$ such that $\langle A, S^g\rangle = X$ and whenever $A \ntrianglelefteq S$, $\langle A, \langle A^{Sg}\rangle\rangle S = X$.*

Received by the editors November 5, 1985 and, in revised form, February 1, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20D05, 20B05.

By Lemma (2.3) in [**T**] and Lemma (2.1) in [**S**] we see that $U_4(q)$, $U_5(q), G_2(q)$, q even, $\hat{A}_6$, and $\mathrm{SL}_2(q)$ satisfy Hypothesis A.

To state our result, we require some standard definitions from the amalgam method. (For the details see [**G**] and §2 below.) Given the amalgam $P \supseteq S \subseteq H$, $G = P *_S H$ acts faithfully on the completion graph $\Gamma = \Gamma(G; P, H)$, and there exist adjacent vertices α and β in Γ such that $G_\alpha = P$, $G_\beta = H$ and $G_{\alpha,\beta} = S$. We set

$$Z_\alpha = \langle \Omega_1(Z(S))\colon S \in \mathrm{Syl}_p(G_\alpha)\rangle$$

and let $b = \min\{d(\lambda, \alpha)\colon Z_\alpha \nsubseteq G_\lambda^{(1)}\}$.

THEOREM. *If $P \supseteq S \subseteq H$ is a pushing-up amalgam for a group satisfying Hypothesis* A, *then $b \leq 4$.*

The point of course is that, in general, $b \leq 4$ along with the group $P/O_p(P)$ always yield a strong bound on the number of noncentral p-chief factors of P in $O_p(P)$. To see the reason below, it is advisable to consult §2 for some definitions. Take $(\alpha, \alpha+1, \ldots, \alpha+5) \subseteq \Gamma$ where $(\alpha, \alpha+4)$ is a critical pair and let $G_{\alpha(i)} = \bigcap_{x \in G_\alpha} G^x_{\alpha,\alpha+i}$. We then see from (3.2.2) and (3.1.4) that $\langle Z^{G_\alpha}_{\alpha+4}\rangle O_p(G_x) = G_\alpha$ and that $\langle Z^{G_\alpha}_{\alpha+4}\rangle$ centralizes $G_{\alpha(5)}$. Hence the noncentral p-chief factors are all contained in $O_p(G_\alpha)/G_{\alpha(4)} = G_{\alpha(2)}/G_{\alpha(4)}$ and $G_{\alpha(4)}/G_{\alpha(5)}$. We now observe that $|O_p(G_\alpha)/G_{\alpha(5)}|$ depends only on $|G_\alpha/O_p(G_\alpha)|$, since $G_{\alpha,\alpha+j} \subseteq G^{(1)}_{\alpha+j}$ for j odd. Furthermore, for each pushing-up group $G_\alpha/O_p(G_\alpha)$, one should then be able to calculate the detail structure of S as is done for example in [**S** and **M**].

One final remark is that our use of the various parts of Hypothesis A is highly modular. Thus if one part of Hypothesis A is replaced by a weaker assumption, then this only entails a change in a very specific part of our argument.

2. Lemmas and notations. Take $P, H, S, G = P *_S H$, and the tree Γ to be as in §1. We let α, β be adjacent vertices in Γ such that $G_\alpha = P$, $G_\beta = H$ and $G_{\alpha\beta} = S$. A list of graph-theoretic notations follows.

(1) $\delta_1 \sim \delta_2$ if δ_1, δ_2 are adjacent vertices in Γ.

(2) $\delta_1 \equiv \delta_2$ if $d(\delta_1, \delta_2)$ is even, where $d(\ ,\ )$ is the usual distance metric on Γ.

(3) $\Delta(\delta) = \{\lambda \in \Gamma\colon \lambda \sim \delta\}$.

(4) Given a vertex λ on a path l in Γ, we write $\lambda + i$ or $\lambda - i$ ($i \in \mathbf{Z}$) to mean the obvious vertices on l distance i from λ.

(5) $G_\delta^{(i)} = \bigcap_{d(\lambda,\delta)\leq i} G_\lambda$.

(6) For $\mu \not\equiv \alpha$, we let $S_\mu = G_{\mu,\varepsilon}$ for any $\varepsilon \sim \mu$.

(7) For any $\delta \equiv \alpha$, we set $Z_\delta = \langle \Omega_1 Z(S)\colon S \in \mathrm{Syl}_p(G_\delta)\rangle$.

(8) $b = \min\{d(\lambda, \alpha)\colon Z_\alpha \nsubseteq G_\lambda^{(1)}\}$.

(9) We say (δ, δ'), where $\delta \equiv \alpha$, is a *critical pair* if $Z_\delta \nsubseteq G_\delta^{(1)}$ and $d(\delta, \delta') = b$.

In the lemmas below, X is a finite group and $S \in \mathrm{Syl}_p(X)$.

(2.1) *If V is a KX-module where* $\operatorname{char} K = p$ *and $V = \langle C_V(S)^X\rangle$, then $V = [V, X]C_V(X)$.*

PROOF. See (1.1) in [**M**]. □

(2.2) *Suppose V is an X-module, $O_p(X) = 1$ and $O^p(X/\phi(X))$ is minimal normal in $X/\phi(X)$. Then either $C_X(V) \subseteq \phi(X)$ or $C_V(S) \subseteq C_V(X)$.*

PROOF. Set L to be the inverse image of $O^p(X/\phi(X))$ in X and $C = C_X(V)$. Then we claim that either $C \subseteq L$ or $L \subseteq C$. For otherwise, we take $T \in \mathrm{Syl}_p(C)$ and observe that by Frattini's argument,

$$X = CN_X(T) = (C_L(V)T)N_X(T) = N_X(T),$$

which violates our hypothesis. Now we note that our claim easily implies the desired conclusion. $\square$

(2.3) *Let $\overline{X} = X/O_p(X)$ and let $\overline{X}/\phi(\overline{X})$ be simple. Let $V = \langle C_V(S)^X \rangle$ be an* FF $\mathrm{GF}(p)\overline{X}$*-module where $\overline{A} \subseteq \overline{X}$ is an offending subgroup. Suppose for all nontrivial irreducible* FF $\mathrm{GF}(p)\overline{X}$*-modules W, $|W\colon C_W(\overline{A})| > |\overline{A}|^{1/2}$. Then $V/C_V(X)$ is an irreducible* FF $\mathrm{GF}(p)\overline{X}$*-module.*

PROOF. Since $\phi(\overline{X})$ is a p'-group, it follows from (2.2) that any noncentral irreducible factor module in V is an FF $\overline{X}$-module. Hence by our hypothesis, V contains a unique noncentral irreducible factor module. Therefore (2.1) yields that $V/C_V(X)$ is an irreducible $\overline{X}$-module. $\square$

(2.4). *Let $Z = \langle \Omega_1(Z(S))^X \rangle$. Suppose $T \leq S$ and $Y \leq X$ such that $\langle T^Y \rangle$ acts transitively on $\{\Omega_1(Z(S))^X\}$ and Y normalizes $[Z,T]$. Then $[[Z,T],T] = [Z,T]$.*

PROOF. Let $L = \langle T^Y \rangle$. Then we observe that

$$Z = [Z,L]\Omega_1(Z(S)).$$

Since $\langle T, Y \rangle$ clearly normalizes $[Z, T^y]$ for all $y \in Y$, we obtain $[Z, \langle T^Y \rangle] = \langle [Z, T^y]\colon y \in Y \rangle = [Z,T]$. Hence $[Z,T] = [[Z,T],T]$. $\square$

3. Proof of Theorem. We use the notation given in §2. Our hypothesis here is that $G_\alpha \supseteq G_{\alpha\beta} \subseteq G_\beta$, $\alpha \sim \beta$, is a pushing-up amalgam for $\overline{G}_\alpha = G_\alpha/O_p(G_\alpha)$ and $\overline{G}_\alpha$ is a group satisfying Hypothesis A. Further, we might as well assume that $B \geq 1$.

(3.1) *The following hold.*
(1) *G_δ acts transitively on $\Delta(\delta)$, for all $\delta \in \Gamma$.*
(2) *$G_{\alpha\beta} = G_\beta^{(1)}$ and $O_p(G_\alpha) = G_\alpha^{(1)} = G_\alpha^{(2)}$.*
(3) *$Z_\alpha \neq \Omega_1(Z(S))$ where $S \in \mathrm{Syl}_p(G_\alpha)$.*
(4) *$Z_\alpha \subseteq Z(O_p(G_\alpha))$.*
(5) *b is even.*
(6) *$O_p(G_\alpha) \in \mathrm{Syl}_p(C_{G_\alpha}(Z_\alpha))$.*
(7) *For any p-group $T \subseteq G_\alpha$, we have $[T, Z_\alpha] = 1$ if and only if $t \subseteq G_\alpha^{(2)}$.*

PROOF. (1), (2), and (3) are immediate consequences of the definition of a pushing-up amalgam. (4) follows from the fact that $b \geq 1$, and (5) is implied by (2).

Since $Z_\alpha = \langle C_{Z_\alpha}(\overline{S})\colon S \in \mathrm{Syl}_p(G_\alpha) \rangle$, from (2.2) and (3) we obtain (6). Lastly, (6) implies (7). $\square$

(3.2) *Let (α, α') be a critical pair. Then the following are true.*

(1) $1 \neq [Z_\alpha, Z_{\alpha'}] \subseteq Z_\alpha \cap Z_{\alpha'}$.

(2) *(α', α) is also a critical pair.*

(3) *$Z_\alpha / C_{Z_\alpha}(G_\alpha)$ is an irreducible* FF $\mathrm{GF}(p)\overline{G}_\alpha$*-module where $\overline{Z}_{\alpha'}$ is a quadratic offending subgroup of $\overline{G}_\alpha$.*

PROOF. (1) and (2) are given by (3.1.7). Now note that, by symmetry, we may assume

$$(*) \qquad |Z_\alpha / Z_\alpha \cap O_p(G_{\alpha'})| \leq |Z_{\alpha'} / Z_{\alpha'} \cap O_p(G_\alpha)|.$$

By (3.1.6), this says that $\overline{Z}_{\alpha'}$ is an offending subgroup of $\overline{G}_\alpha$ acting on Z_α. Therefore, it follows from hypothesis (A.1), (2.3), and (3.2.1) that (3) holds and that we have equality in $(*)$. Hence (3) is true with α and α' switched. □

(3.3) $b \leq 4$.

PROOF. Let us suppose $b \geq 6$. In the context of our proof, whenever $\overline{G}_\delta, \delta \equiv \alpha$, has an unambiguous offending subgroup implicitly given by (3.2.3) we take g_δ to be an element of G_δ which gives the generational property in hypothesis (A.3). We make our argument in several steps.

(1) Let (α, α') be a critical pair and let $\alpha - 1 = \alpha + 1^{g_\alpha}$. If $\delta \in \Delta(\alpha - 1)$ such that $(\delta, \alpha' - 2)$ is not a critical pair, then $Z_\delta Z_\alpha \trianglelefteq G_\alpha$.

Since $b \geq 6$, it follows from (3.1) that $Z_\delta Z_\alpha$ act quadratically on $Z_{\alpha'}$. Hence by (3.2) and hypothesis (A.2), we get that $Z_\delta Z_\alpha O_p(G_{\alpha'}) = Z_\alpha O_p(G_{\alpha'})$. We then observe that this means

$$Z_\delta Z_\alpha \trianglelefteq \langle Z_{\alpha'}, S_{\alpha-1} \rangle.$$

The latter group by hypothesis (A.3) is G_α.

(2) There exists a half-line $l = (\ldots, \alpha - 2, \alpha - 1, \alpha, \ldots, \alpha')$ such that for each $i \in \mathbf{Z}$, $(\alpha + 2i, \alpha + 2i + b) \subseteq l$ is a critical pair and for each $\delta \in l$ with $\delta \equiv \alpha$ and $\delta - 2, \delta + 2 \in l$, we have $\delta + 2^{g_\delta} = \delta - 2$.

Since $\langle Z_\delta \colon \delta \in \Delta(\alpha - 1) \rangle \ntrianglelefteq G_\alpha$, (1) immediately yields through induction a half-line l_1 with the right amount of critical pairs, and such that for all $\mu \in l_1$ with $\mu \equiv \alpha$, $\mu + 1^{g_\mu} = \mu - 1$ where $\mu + 1, \mu - 1 \in l_1$. Now take any critical pair (λ, λ') in l_1 and let $\xi = \lambda + 2^{g_\lambda}$. We claim that $(\xi, \lambda' - 2)$ is also a critical pair. Suppose this is false. Then (1) says that

$$Z_\xi Z_\lambda = Z_{\lambda+2} Z_\lambda.$$

But this means that $Z_{\lambda - b + 2}$ centralizes $Z_{\lambda+2} Z_\lambda$, which contradicts the fact that $(\lambda - b + 2, \lambda + 2)$ is a critical pair. Therefore we may construct l_1 so that it passes through ξ and hence by induction we obtain the existence of the half-line l.

(3) There is no critical pair (λ, λ') in l such that $\overline{Z}_{\lambda'} \trianglelefteq \overline{S}_{\lambda+1}$ in $\overline{G}_\lambda$.

Assume (λ, λ') has the above normality property. Then we note that

$$R = [Z_\lambda, Z_{\lambda'}] = [Z_\lambda, Z_{\lambda'} O_p(G_\lambda)] \trianglelefteq S_{\lambda+1}.$$

Thus as a consequence of hypothesis (A.3), (2), and $b \geq 6$, we get that

$$R \trianglelefteq \langle Z_{\lambda'+2}, S_{\lambda+1} \rangle = G_{\lambda+2},$$

and hence also that $R \trianglelefteq \langle Z_{\lambda'+4}, S_{\lambda+3} \rangle = G_{\lambda+4}$. But this means $R \trianglelefteq G_\lambda$, which is clearly impossible.

(4) The final contradiction.

Take an arbitrary critical pair (λ, λ') in l and again set $R = [Z_\lambda, Z_{\lambda'}]$. Since in $\overline{G}_{\lambda+2}, \overline{Z}_{\lambda'+2} \ntrianglelefteq \overline{X}_{\lambda+3}$, it follows from hypothesis (A.3) and $b \geq 4$ that

$$X_{\lambda+2} = \langle Z_{\lambda'+2}, \langle Z_{\lambda'+2}^{g_{\lambda+2} S_{\lambda+1}} \rangle \rangle$$

centralizes R and acts transitively on $\Delta(\lambda+2)$. For the exact same reason,

$$X_{\lambda+4} = \langle Z_{\lambda'+4}, \langle Z_{\lambda'+4}^{g_{\lambda+4} S_{\lambda+3}} \rangle \rangle$$

is transitive on $\Delta(\lambda+4)$. Now we observe that as $X_{\lambda+2}$ is transitive on $\Delta(\lambda+2)$ and $b \geq 6$, we also know that $X_{\lambda+4}$ centralizes R.

Since $g_{\lambda+2} \in X_{\lambda+2} S_{\lambda+1}$, there exists some $x \in X_{\lambda+2}$ such that $\lambda + 4^x = \lambda$. Therefore, we conclude that R is centralized by some X_λ in G_λ that acts transitively on $\Delta(\alpha)$. This clearly violates (2.4). $\square$

The proof of our theorem is now complete.

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