A NOTE ON PUSHING UP

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Abstract. The amalgam method is used to handle some so-called pushing-up problems in finite groups.

1. Introduction. Generally speaking, pushing-up problems in finite group theory have been posed in terms of obstructions:

Suppose \( S \in \text{Syl}_p(P) \) such that no characteristic subgroup of \( S \) is normal in \( P \). What then can we say about \( S \)? In Goldschmidt's fundamental paper [G], he introduced what is now called the amalgam method. This was later in particular applied to a pushing-up problem by Stellmacher [S] that simplified the works of Niles [N] and Baumann [B]. So let us reformulate our pushing-up question in terms of amalgams. (See [G] for the basic definitions.)

Suppose \( P \) and \( H \) are abstract finite groups. We say \( P \supseteq S \subseteq H \) is a pushing-up amalgam for \( P/O_p(P) \) if

1. \( S \in \text{Syl}_p(P) \).
2. No nontrivial subgroup of \( S \) is normal in \( P \) and \( H \).
3. \( S \triangleleft H \).

Take \( X \) to be a finite group and \( S \in \text{Syl}_p(X) \). Notice that if \( M \) is a maximal \( p \)-local but not a \( p \)-parabolic of \( X \) (i.e., \( N_X(S) \not\subseteq M \)) then one clearly has a pushing-up amalgam. Or if \( N_X(S) \) is contained in a unique maximal \( p \)-local \( M \) of \( X \), then one also has a pushing-up amalgam, unless of course \( M \) is strongly embedded. We further remark that there is some motivation in changing (1) to require simply that \( S \) be a \( p \)-group containing \( O_p(P) \). (See [Gn, p. 280] and [C].) This actually does not entail any essential changes in the argument of our theorem below.

So as a general goal, given any pushing-up amalgam for a \( \text{chev}(p) \)-group one would like to be able to describe \( S \). In this note we give a small indication to be optimistic about solving this problem in general. (See also [M].) Specifically, we consider pushing-up amalgams for \( \text{chev}(p) \)-groups satisfying Hypothesis A below.

First, we define a \( GF(p)X \)-module \( W \) to be an \( FF \)-module if \( C_X(W) \) is a \( p' \)-group and \( X \) has a nontrivial elementary abelian \( p \)-group \( A \) such that \( |W| = |A| \).

HYPOTHESIS A. \( X \) is a group where \( X/O_p(X) \) is simple. For any irreducible \( FF \), \( GF(p)X \)-module \( W \) and any quadratic offending subgroup \( A \) of \( X \), we have

1. \( |W: C_W(A)| = |A| \).
2. \( A \) is a maximal quadratic subgroup of \( X \).
3. If \( A \subseteq S \in \text{Syl}_p(X) \), then there exist \( g \in X \) such that \( \langle A, S^g \rangle = X \) and whenever \( A \nsubseteq S \), \( \langle A, (A^{S^g}) \rangle S = X \).

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By Lemma (2.3) in [T] and Lemma (2.1) in [S] we see that \( U_4(q), U_5(q), G_2(q), q \) even, \( A_6 \), and \( SL_2(q) \) satisfy Hypothesis A.

To state our result, we require some standard definitions from the amalgam method. (For the details see [G] and §2 below.) Given the amalgam \( P \supseteq S \subseteq H \), \( G = P \ast_S H \) acts faithfully on the completion graph \( \Gamma = \Gamma(G; P, H) \), and there exist adjacent vertices \( \alpha \) and \( \beta \) in \( \Gamma \) such that \( G_{\alpha} = P \), \( G_{\beta} = H \) and \( G_{\alpha, \beta} = S \). We set

\[
Z_{\alpha} = \langle \Omega_1(Z(S)) : S \in Syl_p(G_{\alpha}) \rangle
\]

and let \( b = \min\{d(\lambda, \alpha) : Z_{\alpha} \not\subseteq G^{(1)}_{\lambda}\} \).

**Theorem.** If \( P \supseteq S \subseteq H \) is a pushing-up amalgam for a group satisfying Hypothesis A, then \( b \leq 4 \).

The point of course is that, in general, \( b \leq 4 \) along with the group \( P/O_p(P) \) always yield a strong bound on the number of noncentral \( p \)-chief factors of \( P \) in \( O_p(P) \). To see the reason below, it is advisable to consult §2 for some definitions.

Take \( (\alpha, \alpha + 1, \ldots, \alpha + 5) \subseteq \Gamma \) where \( (\alpha, \alpha + 4) \) is a critical pair and let \( G_{\alpha,i} = \bigcap_{S \subseteq G_{\alpha,\alpha+4}} G^{S_{\alpha,i}}_{\alpha,\alpha+4} \). We then see from (3.2.2) and (3.1.4) that \( \langle Z_{G_{\alpha+4}}^{(1)} \rangle_{O_p(G_{\alpha})} = G_{\alpha} \) and that \( \langle Z_{G_{\alpha+4}}^{(1)} \rangle \) centralizes \( G_{\alpha,5} \). Hence the noncentral \( p \)-chief factors are all contained in \( O_p(G_{\alpha})/G_{\alpha,4} ) = G_{\alpha,2}/G_{\alpha,4} \) and \( G_{\alpha,4}/G_{\alpha,5} \). We now observe that \( |O_p(G_{\alpha})/G_{\alpha,5}| \) depends only on \( |G_{\alpha}/O_p(G_{\alpha})| \), since \( G_{\alpha,\alpha+j} \subseteq G^{(1)}_{\alpha+j} \) for \( j \) odd. Furthermore, for each pushing-up group \( G_{\alpha}/O_p(G_{\alpha}) \), one should then be able to calculate the detail structure of \( S \) as is done for example in [S] and [M].

One final remark is that our use of the various parts of Hypothesis A is highly modular. Thus if one part of Hypothesis A is replaced by a weaker assumption, then this only entails a change in a very specific part of our argument.

### 2. Lemmas and notations.

Take \( P, H, S, G = P \ast_S H \), and the tree \( \Gamma \) to be as in §1. We let \( \alpha, \beta \) be adjacent vertices in \( \Gamma \) such that \( G_{\alpha} = P \), \( G_{\beta} = H \) and \( G_{\alpha, \beta} = S \). A list of graph-theoretic notations follows.

1. \( \delta_1 \sim \delta_2 \) if \( \delta_1, \delta_2 \) are adjacent vertices in \( \Gamma \).
2. \( \delta_1 \equiv \delta_2 \) if \( d(\delta_1, \delta_2) \) is even, where \( d( , , ) \) is the usual distance metric on \( \Gamma \).
3. \( \Delta(\delta) = \{ \lambda \in \Gamma : \lambda \sim \delta \} \).
4. Given a vertex \( \lambda \) on a path \( l \) in \( \Gamma \), we write \( \lambda + i \) or \( \lambda - i \) (\( i \in \mathbb{Z} \)) to mean the obvious vertices on \( l \) distance \( i \) from \( \lambda \).
5. \( G^{(i)}_{\delta} = \bigcap_{d(\lambda, \delta) \leq i} G_{\lambda} \).
6. For \( \mu \neq \alpha \), we let \( S_{\mu} = G_{\mu, \varepsilon} \) for any \( \varepsilon \sim \mu \).
7. For any \( \delta \equiv \alpha \), we set \( Z_{\delta} = \langle \Omega_1(Z(S)) : S \in Syl_p(G_{\delta}) \rangle \).
8. \( b = \min\{d(\lambda, \alpha) : Z_{\alpha} \not\subseteq G^{(1)}_{\lambda}\} \).
9. We say \( (\delta, \delta') \), where \( \delta \equiv \alpha \), is a critical pair if \( Z_{\delta} \not\subseteq G^{(1)}_{\delta} \) and \( d(\delta, \delta') = b \). In the lemmas below, \( X \) is a finite group and \( S \in Syl_p(X) \).

(2.1) If \( V \) is a \( KX \)-module where \( \text{char } K = p \) and \( V = \langle C_V(S)^X \rangle \), then \( V = [V, X]C_V(X) \).

**Proof.** See (1.1) in [M]. \( \square \)
(2.2) Suppose $V$ is an $X$-module, $O_p(X) = 1$ and $O^p(X/\phi(X))$ is minimal normal in $X/\phi(X)$. Then either $C_X(V) \subseteq \phi(X)$ or $C_V(S) \subseteq C_V(X)$.

**Proof.** Set $L$ to be the inverse image of $O^p(X/\phi(X))$ in $X$ and $C = C_X(V)$. Then we claim that either $C \subseteq L$ or $L \subseteq C$. For otherwise, we take $T \in \text{Syl}_p(C)$ and observe that by Frattini’s argument,

\[ X = CN_X(T) = (C_L(V)T)N_X(T) = N_X(T), \]

which violates our hypothesis. Now we note that our claim easily implies the desired conclusion. □

(2.3) Let $\overline{X} = X/O_p(X)$ and let $\overline{X}/\phi(\overline{X})$ be simple. Let $V = \langle C(V)S \rangle^X$ be an FF $GF(p)\overline{X}$-module where $\overline{A} \subseteq \overline{X}$ is an offending subgroup. Suppose for all nontrivial irreducible FF $GF(p)\overline{X}$-modules $W$, $|W : C_W(A)| > |A|^{1/2}$. Then $V/C_V(X)$ is an irreducible FF $GF(p)\overline{X}$-module.

**Proof.** Since $\phi(\overline{X})$ is a $p'$-group, it follows from (2.2) that any noncentral irreducible factor module in $V$ is an FF $\overline{X}$-module. Hence by our hypothesis, $V$ contains a unique noncentral irreducible factor module. Therefore (2.1) yields that $V/C_V(X)$ is an irreducible $\overline{X}$-module. □

(2.4). Let $Z = \langle \Omega_1(Z(S)) \rangle^X$. Suppose $T \leq S$ and $Y \leq X$ such that $\langle TY \rangle$ acts transitively on $\{\Omega_1(Z(S))\}^X$ and $Y$ normalizes $[Z,T]$. Then $[[Z,T],T] = [Z,T]$.

**Proof.** Let $L = \langle TY \rangle$. Then we observe that

\[ z = [z,L]^{\Omega_1(Z(S))}. \]


3. **Proof of Theorem.** We use the notation given in §2. Our hypothesis here is that $G_\alpha \supseteq G_{\alpha \beta} \subseteq G_\beta$, $\alpha \sim \beta$, is a pushing-up amalgam for $\overline{G}_\alpha = G_\alpha/O_p(G_\alpha)$ and $\overline{G}_\alpha$ is a group satisfying Hypothesis A. Further, we might as well assume that $B \geq 1$.

(3.1) The following hold.

(1) $G_\delta$ acts transitively on $\Delta(\delta)$, for all $\delta \in \Gamma$.

(2) $G_{\alpha \beta} = G_{\beta}^{(1)}$ and $O_p(G_\alpha) = G_\alpha^{(1)} = G_\alpha^{(2)}$.

(3) $Z_\alpha \neq \Omega_1(Z(S))$ where $S \in \text{Syl}_p(G_\alpha)$.

(4) $Z_\alpha \subseteq Z(O_p(G_\alpha))$.

(5) $b$ is even.

(6) $O_p(G_\alpha) \in \text{Syl}_p(C_{G_\alpha}(Z_\alpha))$.

(7) For any $p$-group $T \leq G_\alpha$, we have $[T,Z_\alpha] = 1$ if and only if $t \subseteq G_\alpha^{(2)}$.

**Proof.** (1), (2), and (3) are immediate consequences of the definition of a pushing-up amalgam. (4) follows from the fact that $b \geq 1$, and (5) is implied by (2).

Since $Z_\alpha = \langle C_{Z_\alpha}(S) : S \in \text{Syl}_p(G_\alpha) \rangle$, from (2.2) and (3) we obtain (6). Lastly, (6) implies (7). □
Let \((\alpha, \alpha')\) be a critical pair. Then the following are true.

1. \(\not\subseteq [Z_{\alpha}, Z_{\alpha'}] \subseteq Z_{\alpha} \cap Z_{\alpha'}\).
2. \((\alpha', \alpha)\) is also a critical pair.
3. \(Z_{\alpha}/C_{\alpha}(G_{\alpha})\) is an irreducible FF GF\((p)\overline{G}_{\alpha}\)-module where \(\overline{Z}_{\alpha'}\) is a quadratic offending subgroup of \(\overline{G}_{\alpha}\).

**Proof.** (1) and (2) are given by (3.1.7). Now note that, by symmetry, we may assume

\[ \frac{Z_{\alpha}}{Z_{\alpha} \cap O_{p}(G_{\alpha'})} \leq \frac{Z_{\alpha'}}{Z_{\alpha'} \cap O_{p}(G_{\alpha})}. \]

By (3.1.6), this says that \(\overline{Z}_{\alpha'}\) is an offending subgroup of \(\overline{G}_{\alpha}\) acting on \(Z_{\alpha}\). Therefore, it follows from hypothesis (A.1), (2.3), and (3.2.1) that (3) holds and that we have equality in (*). Hence (3) is true with \(\alpha\) and \(\alpha'\) switched. \(\square\)

**Proof.** Let us suppose \(b \geq 6\). In the context of our proof, whenever \(\overline{G}_{\delta}, \delta \equiv \alpha,\) has an unambiguous offending subgroup implicitly given by (3.2.3) we take \(g_{\delta}\) to be an element of \(G_{\delta}\) which gives the generational property in hypothesis (A.3). We make our argument in several steps.

1. Let \((\alpha, \alpha')\) be a critical pair and let \(\alpha - 1 = \alpha + 1^{\nu}\). If \(\delta \in \Delta(\alpha - 1)\) such that \((\delta, \alpha' - 2)\) is not a critical pair, then \(Z_{\delta}Z_{\alpha} \leq G_{\alpha}\).

Since \(b \geq 6\), it follows from (3.1) that \(Z_{\delta}Z_{\alpha}\) act quadratically on \(Z_{\alpha'}\). Hence by (3.2) and hypothesis (A.2), we get that \(Z_{\delta}Z_{\alpha}O_{p}(G_{\alpha'}) = Z_{\alpha}O_{p}(G_{\alpha'})\). We then observe that this means

\[ \frac{Z_{\delta}Z_{\alpha}}{Z_{\alpha} \cap O_{p}(G_{\alpha'})} = \frac{Z_{\alpha} \cap O_{p}(G_{\alpha'})}{Z_{\alpha} \cap O_{p}(G_{\alpha'})}. \]

The latter group by hypothesis (A.3) is \(G_{\alpha}\).

2. There exists a half-line \(l = (\ldots, \alpha - 2, \alpha - 1, \alpha, \ldots, \alpha')\) such that for each \(i \in Z\), \((\alpha + 2i, \alpha + 2i + b) \subseteq l\) is a critical pair and for each \(\delta \in l\) with \(\delta \equiv \alpha\) and \(\delta - 2, \delta + 2 \in l\), we have \(\delta + 2^{\nu} = \delta - 2\).

Since \(\langle Z_{\delta} : \delta \in \Delta(\alpha - 1)\rangle \not\subseteq G_{\alpha}\), (1) immediately yields through induction a half-line \(l_{1}\) with the right amount of critical pairs, and such that for all \(\mu \in l_{1}\) with \(\mu \equiv \alpha, \mu + 1^{\nu} = \mu - 1\) where \(\mu + 1, \mu - 1 \in l_{1}\). Now take any critical pair \((\lambda, \lambda')\) in \(l_{1}\) and let \(\xi = \lambda + 2^{\nu}\). We claim that \((\xi, \lambda' - 2)\) is also a critical pair. Suppose this is false. Then (1) says that

\[ Z_{\delta}Z_{\lambda} = Z_{\lambda + 2}Z_{\lambda}. \]

But this means that \(Z_{\lambda - b + 2}\) centralizes \(Z_{\lambda + 2}Z_{\lambda}\), which contradicts the fact that \((\lambda - b + 2, \lambda + 2)\) is a critical pair. Therefore we may construct \(l_{1}\) so that it passes through \(\xi\) and hence by induction we obtain the existence of the half-line \(l\).

3. There is no critical pair \((\lambda, \lambda')\) in \(l\) such that \(\overline{Z}_{\lambda'} \leq S_{\lambda + 1}\) in \(\overline{G}_{\lambda}\).

Assume \((\lambda, \lambda')\) has the above normality property. Then we note that

\[ R = [Z_{\lambda}, Z_{\lambda'}] = [Z_{\lambda}, Z_{\lambda'}O_{p}(G_{\lambda})] \leq S_{\lambda + 1}. \]

Thus as a consequence of hypothesis (A.3), (2), and \(b \geq 6\), we get that

\[ R \leq \langle Z_{\lambda' + 2}, S_{\lambda + 1} \rangle = G_{\lambda + 2}, \]

and hence also that \(R \leq \langle Z_{\lambda' + 4}, S_{\lambda + 3} \rangle = G_{\lambda + 4}\). But this means \(R \leq G_{\lambda}\), which is clearly impossible.
(4) The final contradiction. Take an arbitrary critical pair \((\lambda, \lambda')\) in \(l\) and again set \(R = [Z_{\lambda}, Z_{\lambda'}]\). Since in \(G_{\lambda+2}, Z_{\lambda'+2} \not\subseteq X_{\lambda+3}\), it follows from hypothesis (A.3) and \(b \geq 4\) that

\[X_{\lambda+2} = \langle Z_{\lambda'+2}, \langle Z_{\lambda'+2}^{g_{\lambda+2} S_{\lambda+1}} \rangle \rangle\]

centralizes \(R\) and acts transitively on \(\Delta(\lambda + 2)\). For the exact same reason,

\[X_{\lambda+4} = \langle Z_{\lambda'+4}, \langle Z_{\lambda'+4}^{g_{\lambda+4} S_{\lambda+3}} \rangle \rangle\]

is transitive on \(\Delta(\lambda + 4)\). Now we observe that as \(X_{\lambda+2}\) is transitive on \(\Delta(\lambda + 2)\) and \(b \geq 6\), we also know that \(X_{\lambda+4}\) centralizes \(R\).

Since \(g_{\lambda+2} \in X_{\lambda+2} S_{\lambda+1}\), there exists some \(x \in X_{\lambda+2}\) such that \(\lambda + 4^x = \lambda\). Therefore, we conclude that \(R\) is centralized by some \(X_{\lambda}\) in \(G_{\lambda}\) that acts transitively on \(\Delta(\lambda)\). This clearly violates (2.4). \(\Box\)

The proof of our theorem is now complete.

REFERENCES


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