A NOTE ON PUSHING UP

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ABSTRACT. The amalgam method is used to handle some so-called pushing-up problems in finite groups.

1. Introduction. Generally speaking, pushing-up problems in finite group theory have been posed in terms of obstructions:

Suppose $S \leq P$ such that no characteristic subgroup of $S$ is normal in $P$. What then can we say about $S$? In Goldschmidt's fundamental paper [G], he introduced what is now called the amalgam method. This was later in particular applied to a pushing-up problem by Stellmacher [S] that simplified the works of Niles [N] and Baumann [B]. So let us reformulate our pushing-up question in terms of amalgams. (See [G] for the basic definitions.)

Suppose $P$ and $H$ are abstract finite groups. We say $P \leq S \leq H$ is a pushing-up amalgam for $P/O_p(P)$ if

1. $S \in \text{Syl}_p(P)$.
2. No nontrivial subgroup of $S$ is normal in $P$ and $H$.
3. $S \leq H$.

Take $X$ to be a finite group and $S \in \text{Syl}_p(X)$. Notice that if $M$ is a maximal $p$-local but not a $p$-parabolic of $X$ (i.e., $N_X(S) \not\subseteq M$) then one clearly has a pushing-up amalgam. Or if $N_X(S)$ is contained in a unique maximal $p$-local $M$ of $X$, then one also has a pushing-up amalgam, unless of course $M$ is strongly embedded. We further remark that there is some motivation in changing (1) to require simply that $S$ be a $p$-group containing $O_p(P)$. (See [Gn, p. 280] and [C].) This actually does not entail any essential changes in the argument of our theorem below.

So as a general goal, given any pushing-up amalgam for a chev($p$)-group one would like to be able to describe $S$. In this note we give a small indication to be optimistic about solving this problem in general. (See also [M].) Specifically, we consider pushing-up amalgams for chev($p$)-groups satisfying Hypothesis A below.

First, we define a GF($p$)$X$-module $W$ to be an FF-module if $C_X(W)$ is a $p'$-group and $X$ has a nontrivial elementary abelian $p$-group $A$ such that $|W| = |C_W(A)|$. $A$ is called an offending subgroup.

HYPOTHESIS A. $X$ is a group where $X/\phi(X)$ is simple. For any irreducible FF, GF($p$)$X$-module $W$ and any quadratic offending subgroup $A$ of $X$, we have

1. $|W : C_W(A)| = |A|$. 
2. $A$ is a maximal quadratic subgroup of $X$.
3. If $A \leq S \in \text{Syl}_p(X)$, then there exist $g \in X$ such that $\langle A, S^g \rangle = X$ and whenever $A \not\subseteq S$, $\langle A, (A^S)^g \rangle S = X$.

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By Lemma (2.3) in [T] and Lemma (2.1) in [S] we see that $U_4(q), U_5(q), G_2(q),$ $q$ even, $A_6,$ and $SL_2(q)$ satisfy Hypothesis A.

To state our result, we require some standard definitions from the amalgam method. (For the details see [G] and §2 below.) Given the amalgam $P \supset S \subseteq H,$ $G = P \ast_S H$ acts faithfully on the completion graph $\Gamma = \Gamma(G; P, H),$ and there exist adjacent vertices $\alpha$ and $\beta$ in $\Gamma$ such that $G_{\alpha} = P,$ $G_{\beta} = H$ and $G_{\alpha,\beta} = S.$ We set

$$Z_\alpha = \langle \Omega_1(Z(S)) : S \in Syl_p(G_\alpha) \rangle$$

and let $b = \min\{d(\lambda, \alpha) : Z_\alpha \not\subseteq G_\lambda^{(1)}\}.$

**Theorem.** If $P \supset S \subseteq H$ is a pushing-up amalgam for a group satisfying Hypothesis A, then $b \leq 4.$

The point of course is that, in general, $b \leq 4$ along with the group $P/O_p(P)$ always yield a strong bound on the number of noncentral $p$-chief factors of $P$ in $O_p(P).$ To see the reason below, it is advisable to consult §2 for some definitions. Take $(\alpha, \alpha + 1, \ldots, \alpha + 6) \subseteq \Gamma$ where $(\alpha, \alpha + 1)$ is a critical pair and let $G_{\alpha(i)} = \bigcap_{s \in G_\alpha} G_{\alpha,\alpha+i}^{(s)}.$ We then see from (3.2.2) and (3.1.4) that $\langle Z_{\alpha+4}^{G_\alpha} \rangle O_p(G_\alpha) = G_\alpha$ and that $\langle Z_{\alpha+4}^{G_\alpha} \rangle$ centralizes $G_{\alpha(5)}.$ Hence the noncentral $p$-chief factors are all contained in $O_p(G_\alpha)/G_{\alpha(4)} = G_{\alpha(2)}/G_{\alpha(4)}$ and $G_{\alpha(4)}/G_{\alpha(5)}.$ We now observe that $|O_p(G_\alpha)/G_{\alpha(5)}|$ depends only on $|G_\alpha/O_p(G_\alpha)|,$ since $G_{\alpha,\alpha+j} \subseteq G_{\alpha,j}^{(1)}$ for $j$ odd. Furthermore, for each pushing-up group $G_\alpha/O_p(G_\alpha),$ one should then be able to calculate the detail structure of $S$ as is done for example in [S and M].

One final remark is that our use of the various parts of Hypothesis A is highly modular. Thus if one part of Hypothesis A is replaced by a weaker assumption, then this only entails a change in a very specific part of our argument.

2. Lemmas and notations. Take $P, H, S, G = P \ast_S H,$ and the tree $\Gamma$ to be as in §1. We let $\alpha, \beta$ be adjacent vertices in $\Gamma$ such that $G_\alpha = P,$ $G_\beta = H$ and $G_{\alpha,\beta} = S.$ A list of graph-theoretic notations follows.

1. $\delta_1 \sim \delta_2$ if $\delta_1, \delta_2$ are adjacent vertices in $\Gamma.$
2. $\delta_1 \equiv \delta_2$ if $d(\delta_1, \delta_2)$ is even, where $d(\delta_1, \delta_2)$ is the usual distance metric on $\Gamma.$
3. $\Delta(\delta) = \{\lambda \in \Gamma : \delta \sim \lambda\}.$
4. Given a vertex $\lambda$ on a path $l$ in $\Gamma,$ we write $\lambda + i$ or $\lambda - i$ ($i \in \mathbb{Z}$) to mean the obvious vertices on $l$ distance $i$ from $\lambda.$
5. $G^{(i)}_\delta = \bigcap_{d(\lambda, \delta) \leq i} G_\lambda.$
6. For $\mu \neq \alpha,$ we let $S_\mu = G_{\mu,\epsilon}$ for any $\epsilon \sim \mu.$
7. For any $\delta \equiv \alpha,$ we set $Z_\delta = \langle \Omega_1(Z(S)) : S \in Syl_p(G_\delta) \rangle.$
8. $b = \min\{d(\lambda, \alpha) : Z_{\alpha} \not\subseteq G_\lambda^{(1)}\}.$
9. We say $(\delta, \delta'),$ where $\delta \equiv \alpha,$ is a critical pair if $Z_\delta \not\subseteq G_\delta^{(1)}$ and $d(\delta, \delta') = b.$

In the lemmas below, $X$ is a finite group and $S \in Syl_p(X).$

(2.1) If $V$ is a $KX$-module where char $K = p$ and $V = \langle C_V(S)^X \rangle,$ then $V = [V, X]C_V(X).$

**Proof.** See (1.1) in [M]. □
Suppose $V$ is an $X$-module, $O_p(X) = 1$ and $O_p(X/\phi(X))$ is minimal normal in $X/\phi(X)$. Then either $C_X(V) \subseteq \phi(X)$ or $C_V(S) \subseteq C_V(X)$.

**Proof.** Set $L$ to be the inverse image of $O_p(X/\phi(X))$ in $X$ and $G = C_X(V)$. Then we claim that either $C \subseteq L$ or $L \subseteq C$. For otherwise, we take $T \in Syl_p(C)$ and observe that by Frattini's argument,

$$X = CN_X(T) = (C_L(V)T)N_X(T) = N_X(T),$$

which violates our hypothesis. Now we note that our claim easily implies the desired conclusion. □

(2.3) Let $\overline{X} = X/O_p(X)$ and let $\overline{X}/\phi(\overline{X})$ be simple. Let $V = \langle C_V(S)^X \rangle$ be an FF $GF(p)^X$-module where $A \subseteq \overline{X}$ is an offending subgroup. Suppose for all nontrivial irreducible FF $GF(p)^X$-modules $W$, $|W : C_W(A)| > |A|^{1/2}$. Then $V/C_V(X)$ is an irreducible FF $GF(p)^X$-module.

**Proof.** Since $\phi(\overline{X})$ is a $p'$-group, it follows from (2.2) that any noncentral irreducible factor module in $V$ is an FF $X$-module. Hence by our hypothesis, $V$ contains a unique noncentral irreducible factor module. Therefore (2.1) yields that $V/C_V(X)$ is an irreducible $\overline{X}$-module. □

(2.4) Let $Z = \langle \Omega_1(Z(S))^X \rangle$. Suppose $T \leq S$ and $Y \leq X$ such that $\langle T^Y \rangle$ acts transitively on $\{\Omega_1(Z(S))^X\}$ and $Y$ normalizes $[Z,T]$. Then $[[Z,T],T] = [Z,T]$.

**Proof.** Let $L = \langle T^Y \rangle$. Then we observe that

$$Z = [Z,L]\Omega_1(Z(S)).$$


3. **Proof of Theorem.** We use the notation given in §2. Our hypothesis here is that $G_\alpha \supseteq G_{\alpha\beta} \subseteq G_\beta$, $\alpha \sim \beta$, is a pushing-up amalgam for $\overline{G}_\alpha = G_\alpha/O_p(G_\alpha)$ and $\overline{G}_\alpha$ is a group satisfying Hypothesis A. Further, we might as well assume that $B \geq 1$.

(3.1) The following hold.

1. $G_\delta$ acts transitively on $\Delta(\delta)$, for all $\delta \in \Gamma$.
2. $G_{\alpha\beta} = G_{\beta}^{(1)}$ and $O_p(G_\alpha) = G_\alpha^{(1)} = G_\alpha^{(2)}$.
3. $Z_\alpha \neq \Omega_1(Z(S))$ where $S \in Syl_p(G_\alpha)$.
4. $Z_\alpha \subseteq Z(O_p(G_\alpha))$.
5. $b$ is even.
6. $O_p(G_\alpha) \in Syl_p(C_{G_\alpha}(Z_\alpha))$.
7. For any $p$-group $T \subseteq G_\alpha$, we have $[T,Z_\alpha] = 1$ if and only if $t \subseteq G_\alpha^{(2)}$.

**Proof.** (1), (2), and (3) are immediate consequences of the definition of a pushing-up amalgam. (4) follows from the fact that $b \geq 1$, and (5) is implied by (2).

Since $Z_\alpha = \langle C_{Z_\alpha}(S) : S \in Syl_p(G_\alpha) \rangle$, from (2.2) and (3) we obtain (6). Lastly, (6) implies (7). □
(3.2) Let \((\alpha, \alpha')\) be a critical pair. Then the following are true.
(1) \(1 \neq [Z_{\alpha}, Z_{\alpha'}] \subseteq Z_{\alpha} \cap Z_{\alpha'}\).
(2) \((\alpha', \alpha)\) is also a critical pair.
(3) \(Z_{\alpha}/C_{\alpha}(G_{\alpha})\) is an irreducible FF GF\((p)\overline{G}_{\alpha}\)-module where \(\overline{Z}_{\alpha}\) is a quadratic offending subgroup of \(\overline{G}_{\alpha}\).

PROOF. (1) and (2) are given by (3.1.7). Now note that, by symmetry, we may assume
\[
(*) \quad |Z_{\alpha}/Z_{\alpha} \cap O_{p}(G_{\alpha'})| \leq |Z_{\alpha'}/Z_{\alpha'} \cap O_{p}(G_{\alpha})|.
\]
By (3.1.6), this says that \(\overline{Z}_{\alpha'}\) is an offending subgroup of \(\overline{G}_{\alpha}\) acting on \(Z_{\alpha}\). Therefore, it follows from hypothesis (A.1), (2.3), and (3.2.1) that (3) holds and that we have equality in (*). Hence (3) is true with \(a\) and \(a'\) switched. □

(3.3) \(b \leq 4\).

PROOF. Let us suppose \(b \geq 6\). In the context of our proof, whenever \(G_{\delta}, \delta \equiv \alpha\), has an unambiguous offending subgroup implicitly given by (3.2.3) we take \(g_{\delta}\) to be an element of \(G_{\delta}\) which gives the generational property in hypothesis (A.3). We make our argument in several steps.

(1) Let \((\alpha, \alpha')\) be a critical pair and let \(\alpha - 1 = \alpha + 1^{2\alpha}\). If \(\delta \in \Delta(\alpha - 1)\) such that \((\delta, \alpha' - 2)\) is not a critical pair, then \(Z_{\delta}Z_{\alpha} \subseteq G_{\alpha}\).

Since \(b \geq 6\), it follows from (3.1) that \(Z_{\delta}Z_{\alpha}\) act quadratically on \(Z_{\alpha}\). Hence by (3.2) and hypothesis (A.2), we get that \(Z_{\delta}Z_{\alpha}O_{p}(G_{\alpha'}) = Z_{\alpha}O_{p}(G_{\alpha'})\). We then observe that this means
\[
Z_{\delta}Z_{\alpha} \leq (Z_{\alpha'}, S_{\alpha - 1}).
\]
The latter group by hypothesis (A.3) is \(G_{\alpha}\).

(2) There exists a half-line \(l = (\ldots, \alpha - 2, \alpha - 1, \alpha, \ldots, \alpha')\) such that for each \(i \in \mathbb{Z}\), \((\alpha + 2i, \alpha + 2i + b) \subseteq l\) is a critical pair and for each \(\delta \in l\) with \(\delta \equiv \alpha\) and \(\delta - 2, \delta + 2 \in l\), we have \(\delta + 2^{\delta} = \delta - 2\).

Since \(\langle Z_{\delta}: \delta \in \Delta(\alpha - 1)\rangle \not\subseteq G_{\alpha}\), (1) immediately yields through induction a half-line \(l_{1}\) with the right amount of critical pairs, and such that for all \(\mu \in l_{1}\) with \(\mu \equiv \alpha\), \(\mu + 1^{2\mu} = \mu - 1\) where \(\mu + 1, \mu - 1 \in l_{1}\). Now take any critical pair \((\lambda, \lambda')\) in \(l_{1}\) and let \(\xi = \lambda + 2^{2\lambda}\). We claim that \((\xi, \lambda' - 2)\) is also a critical pair. Suppose this is false. Then (1) says that
\[
Z_{\xi}Z_{\lambda} = Z_{\lambda + 2}Z_{\lambda}.
\]
But this means that \(Z_{\lambda - b + 2}\) centralizes \(Z_{\lambda + 2}Z_{\lambda}\), which contradicts the fact that \((\lambda - b + 2, \lambda + 2)\) is a critical pair. Therefore we may construct \(l_{1}\) so that it passes through \(\xi\) and hence by induction we obtain the existence of the half-line \(l\).

(3) There is no critical pair \((\lambda, \lambda')\) in \(l\) such that \(Z_{\lambda'} \leq S_{\lambda + 1}\) in \(\overline{G}_{\lambda}\).

Assume \((\lambda, \lambda')\) has the above normality property. Then we note that
\[
R = [Z_{\lambda}, Z_{\lambda}] = [Z_{\lambda}, Z_{\lambda}O_{p}(G_{\lambda})] \leq S_{\lambda + 1}.
\]
Thus as a consequence of hypothesis (A.3), (2), and \(b \geq 6\), we get that
\[
R \leq \langle Z_{\lambda' + 4}, S_{\lambda + 1}\rangle = G_{\lambda + 2},
\]
and hence also that \(R \leq \langle Z_{\lambda' + 4}, S_{\lambda + 3}\rangle = G_{\lambda + 4}\). But this means \(R \leq G_{\lambda}\), which is clearly impossible.
(4) The final contradiction.

Take an arbitrary critical pair \((\lambda, \lambda')\) in \(l\) and again set \(R = [Z_\lambda, Z_{\lambda'}]\). Since in 
\(G_{\lambda+2}, Z_{\lambda'}+2 \not\subseteq X_{\lambda+3}\), it follows from hypothesis (A.3) and \(b \geq 4\) that

\[
X_{\lambda+2} = \langle Z_{\lambda'+2}, \langle Z_{\lambda'+2}^{g_{\lambda+2}}S_{\lambda+1}^{i} \rangle \rangle
\]

centralizes \(R\) and acts transitively on \(\Delta(\lambda + 2)\). For the exact same reason,

\[
X_{\lambda+4} = \langle Z_{\lambda'+4}, \langle Z_{\lambda'+4}^{g_{\lambda+4}}S_{\lambda+3}^{i} \rangle \rangle
\]
is transitive on \(\Delta(\lambda + 4)\). Now we observe that as \(X_{\lambda+2}\) is transitive on \(\Delta(\lambda + 2)\) and \(b \geq 6\), we also know that \(X_{\lambda+4}\) centralizes \(R\).

Since \(g_{\lambda+2} \in X_{\lambda+2}S_{\lambda+1}\), there exists some \(x \in X_{\lambda+2}\) such that \(\lambda + 4x = \lambda\). Therefore, we conclude that \(R\) is centralized by some \(X_\lambda\) in \(G_\lambda\) that acts transitively on \(\Delta(\alpha)\). This clearly violates (2.4). \(\blacksquare\)

The proof of our theorem is now complete.

REFERENCES


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