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EMBEDDING COUNTABLE RINGS IN 2-GENERATOR RINGS

K. C. O'MEARA

ABSTRACT. A short elementary proof involving matrices is used to show that any countable ring can be embedded in a 2-generator ring. Immediate corollaries are the known results that any countable (respectively finite) semigroup can be embedded in a 2-generator (respectively finite 2-generator) semigroup.

The object of this note is to show that any countable ring can be embedded in a 2-generator ring, using a short elementary proof involving matrices. An immediate corollary of our approach is the known result that any countable semigroup can be embedded in a 2-generator semigroup. The latter was first established by Evans [**1**] in 1952 using free semigroups, and since then a number of other proofs have been given (e.g. [**3**, **4**, **5**]). Interest in these embedding theorems was sparked by the famous Higman, Neumann, and Neumann paper [**2**] in 1949, which used free products to show that any countable group can be embedded in a 2-generator group. That this type of result does not hold in all algebraic systems is shown by the fact that the only countable abelian groups which can be embedded in 2-generator abelian groups are those which are the direct sum of two cyclic groups.

For a ring R we let R_∞ be the ring of all countably-infinite, column-finite matrices over R (that is, $\aleph_0 \times \aleph_0$ matrices with only a finite number of nonzero entries in each column, and with the usual matrix addition and multiplication). A ring need not have an identity. Note, however, that if R has an identity so also does R_∞.

THEOREM. *Given any countable ring R, there exists a ring T and a ring embedding $\theta\colon R \to T$ such that $\theta(R)$ is contained in a 2-generator subsemigroup of the multiplicative semigroup of T.*

PROOF. *Step* 1. *Embedding R in a* 3*-generator subsemigroup.* We can assume R has an identity because the standard construction for adding one preserves countability. Suppose $R = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let $S = R_\infty$ and consider the ring embedding

$$\psi\colon R \to S, \qquad \psi(r) = \begin{pmatrix} r & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \end{pmatrix}.$$

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Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & \cdots \\ & & \text{\Large 0} & & \\ & & & & \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix},$$

each a member of R_∞. Observe that right multiplication of a matrix by β shifts the columns one step to the left, while right multiplication by γ leaves the first column unchanged and sets all other columns to 0. Hence

$$\psi(a_n) = \begin{pmatrix} a_n & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} = \alpha\beta^{n-1}\gamma$$

for all n, provided we interpret $\alpha\beta^0\gamma$ as $\alpha\gamma$. This shows $\psi(R) \subseteq \langle \alpha, \beta, \gamma \rangle$, where $\langle \alpha, \beta, \gamma \rangle$ is the multiplicative subsemigroup of S generated by α, β, γ.

Step 2. *Going from* 3 (*or* $n < \infty$) *generators to* 2. Let $T = S_4$ be the ring of 4×4 matrices over S, and again consider the "corner" embedding

$$\phi\colon S \to T, \qquad \phi(s) = \begin{pmatrix} s & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \in S_4.$$

Let

$$\eta = \begin{pmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} \alpha & \beta & \gamma & 1 \\ & \text{\Large 0} & & \\ & & & \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \in T.$$

By noting the effect of right multiplication by η and δ (similar to β and γ) we have

$$\xi\eta^3 = \delta \in \langle \eta, \xi \rangle$$

and

$$\phi(\alpha) = \begin{pmatrix} \alpha & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \xi\delta, \quad \phi(\beta) = \begin{pmatrix} \beta & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \xi\eta\delta,$$

$$\phi(\gamma) = \begin{pmatrix} \gamma & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \xi\eta^2\delta.$$

Thus $\phi(\psi(R)) \subseteq \langle \eta, \xi \rangle$, whence $\theta = \phi \circ \psi \colon R \to T$ is the desired ring embedding. The proof is complete. □

Presented with any countable semigroup S, we can always embed S in the multiplicative semigroup of some countable ring R (for example, the semigroup ring $\mathbf{Z}[S]$). Hence we have

COROLLARY. *Every countable ring can be embedded in a 2-generator ring. Every countable semigroup can be embedded in a 2-generator semigroup.* □

There are two further bonuses of our proof, for it shows:

(1) Every countable-dimensional algebra over a field can be embedded in a 2-generator algebra (in Step 1 take $\{a_1, a_2, \ldots, a_n, \ldots\}$ to be a basis for R).

(2) Every finite ring (respectively finite semigroup) can be embedded in a 2-generator finite ring (respectively 2-generator finite semigroup). This follows from Step 2 alone, for a general n, using the obvious $(n+1) \times (n+1)$ matrices. The semigroup version of this was obtained by Neumann [**3**].

ADDED IN PROOF. The author has recently learned that the above Corollary for rings and the result (1) were obtained by A. I. Mal′tsev in *A representation of nonassociative rings*, Uspehi. Mat. Nauk **7** (1955), 181–185. Mal′tsev's proof uses free rings and is similar to Evans' proof [**1**] for semigroups. Also using free rings, V. Ya. Belyaev has shown in *Subrings of finitely presented associative rings*, Algebra i Logika **17** (1978), 627–638, that any (associative) ring with a recursively enumerable set of defining relations can be embedded in a 2-generator finitely presented ring.

The techniques of the present paper can be modified to show that any countable (respectively finite) ring with identity can be embedded in a 2-generator (respectively finite 2-generator) ring with identity such that the embedding preserves the identity and respects the centers. (Unlike the Theorem, however, the embedding is no longer into a 2-generator multiplicative subsemigroup.) This will appear in a paper *Identity-preserving embeddings of countable rings into 2-generator rings* by the author, C. E. Vinsonhaler, and W. J. Wickless. The embeddings used by Mal′tsev and Belyaev (above) do not preserve the identity.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CANTERBURY, CHRISTCHURCH 1, NEW ZEALAND