TWO-GENERATOR GROUPS
WITH PERFECT FRATTINI SUBGROUPS

M. J. EVANS

ABSTRACT. In this paper, 2-generator groups with perfect Frattini subgroups are constructed. The construction also shows that every countable group can be embedded in the Frattini subgroup of a 2-generator group.

1. Introduction. It is well-known that the Frattini subgroup $\Phi(G)$ of a finite group $G$ is nilpotent. However, for infinite groups the situation is considerably more complicated. For instance, in [1] P. Hall constructed infinite 2-generator soluble groups with non-nilpotent Frattini subgroups. More recently, M. F. Newman (unpublished) has constructed finitely generated, infinite $p$-groups with insoluble Frattini subgroups.

In this note we use a construction similar to Hall's (loc. cit.) to obtain 2-generator groups with perfect Frattini subgroups. More precisely, we prove

THEOREM 1.1. Let $K$ be a countable perfect group. Then there exists a 2-generator group $G_K$, such that $\Phi(G_K)$ is a direct product of $\aleph_0$ copies of $K$.

Since each countable group can be embedded in a countable, nonabelian, simple (and therefore perfect) group (e.g. [2, Theorem 3.4]), Theorem 1.1 implies the following.

COROLLARY 1.2. Every countable group can be embedded in the Frattini subgroup of a 2-generator group.

2. The construction. Throughout the sequel $M$ denotes a countable perfect group of infinite exponent.

Suppose $M = \{m_1, m_2, m_3, \ldots \}$ and let $S$ denote the sequence $m_1, m_2, m_1, m_2, m_3, m_4, m_1, m_2, m_3, m_4, m_1, \ldots, m_i, m_1, \ldots, m_i, m_{i+1}, m_1, \ldots$. Write $S_j$ for the $j$th term of $S$.

REMARK. Note that for any integer $N$ and any $m \in M$ there exists an integer $n > N$ such that $S_n = m$. This fact will be crucial in the proof of Lemma 3.1.

Let $C = \prod_{i \in \mathbb{Z}} M_i$ and $H = \prod_{i \in \mathbb{Z}} M_i$, where $M_i \cong M$ for all $i \in \mathbb{Z}$. Thus $C$ is the cartesian product and $H$ the direct product of $\aleph_0$ copies of $M$. Given any $g \in C$ we write $(g)_i$ for the $i$th coordinate of $g$ and define the support of $g$ to be the set $\sigma(g) = \{i | (g)_i \neq 1\}$. Clearly $g \in H$ if and only if $\sigma(g)$ is a finite set.

Let $z$ generate an infinite cyclic group that is disjoint from $C$ and define an action of $\langle z \rangle$ on $C$ by

$$(g^z)_i = (g)_i, \text{ for all } g \in C \text{ and } i \in \mathbb{Z}.$$ 

Define $C^*$ to be the split extension of $C$ by $\langle z \rangle$. 

Received by the editors March 6, 1986.
Let $w \in C$ be the element such that
\[
(w)_{2j} = S_j \quad \text{if } j \geq 1, \\
(w)_l = 1 \quad \text{if } l \neq 2^j \text{ for all } j \geq 1.
\]

Finally, define $G_M = \langle H, w, z \rangle$, a subgroup of $C^\ast$. Observe that $M_i^w = M_{i+1}$ and $M_i^z = M_i$, for all $i \in \mathbb{Z}$, so $H < G_M$.

The notation introduced above will be used for the remainder of this note. We shall show that $\Phi(G_M) = H$ and use this fact to deduce Theorem 1.1.

3. The lemmas. Our main lemma is the following.

**Lemma 3.1.** Let $h_1$ and $h_2$ be arbitrary elements of $H$. Then $G_M = \langle h_1w, h_2z \rangle$.

**Proof.** Since $\sigma(h_1)$ and $\sigma(h_2)$ are finite sets, there exists an integer $N > 0$ such that $(h_1w)_n = (w)_n$ and $(h_2)_n = 1$ for all $n \geq N$. We consider such an $N$ fixed once for all and define $p$ to be the least integer such that $2^p \geq N$. We shall prove that $M_{2^p} \leq \langle h_1w, h_2z \rangle$. Since $(h_2z)^{-k}M_{2^p}(h_2z)^k = M_{2^p+k}$, for all $k \in \mathbb{Z}$, and $H = \bigcup_{i \in \mathbb{Z}} M_i$, the result then follows immediately.

As $M$ is perfect, it suffices to show that, given arbitrary elements $m_k, m_j \in M$, there exists $y \in \langle h_1w, h_2z \rangle \cap H$ such that

\begin{align*}
(1) & \quad (y)_{2^p} = [m_k, m_j] \\
(2) & \quad (y)_l = 1 \quad \text{if } l \neq 2^p.
\end{align*}

Suppose that $m_k$ and $m_j$ are given elements of $M$ and let $t$ denote the least integer in $\sigma(h_1w)$. It follows from the remark in §2 that there exists an integer $r > p$ such that $S_r = m_k$. Furthermore, there exists an integer $u > r$ such that $S_u = m_j$ and $2^r - t < 2^u - 1$. The letters $r$ and $u$ retain this meaning throughout.

Define $\lambda_1, \lambda_2 \in \langle h_1w, h_2z \rangle$ by $\lambda_i = (h_2z)^{q_i}h_1w(h_2z)^{-q_i}$, where $q_1 = 2^r - 2^p$ and $q_2 = 2^u - 2^p$. We claim that $y = [\lambda_1, \lambda_2]$ is the desired element satisfying (1) and (2).

Bearing in mind that $q_i > 0$, for $i = 1, 2$, we have
\[
(h_2z)^{q_i} = h_2h_2^{-1} \cdots h_2^{-q_i}z^{q_i} = f_i z^{q_i},
\]
where $f_i \in H$. Thus $(\lambda_i)_{2^p} = (f_i)_{2^p}((h_1w)^{z^{-q_i}})_{2^p}(f_i^{-1})_{2^p}$. However, $f_i = \prod_{a=0}^{q_i-1} h_2^{-a}$ so
\[
(f_i)_{2^p} = \prod_{a=0}^{q_i-1} (h_2^{-a})_{2^p} = \prod_{a=0}^{q_i-1} (h_2)_{2^p+a}
\]
for $i = 1, 2$. Now $2^p + a \geq N$ for $a = 0, 1, \ldots, q_i - 1$ and, by our choice of $N$, $(h_2)_{2^p} = (h_2)_1 = 1$ for all $n \geq N$. Hence $(f_i)_{2^p} = 1$ for $i = 1, 2$. Therefore $(\lambda_1)_{2^p} = ((h_1w)^{z^{-q_i}})_{2^p} = (h_1w)_{2^p+q_i}$ for $i = 1, 2$. Recall that $(h_1w)_n = (w)_n$ for all $n \geq N$. It follows that $(\lambda_1)_{2^p} = (h_1w)_{2^r} = (w)_{2^r} = S_r = m_k$ by our choice of $r$, and similarly $(\lambda_2)_{2^p} = (h_1w)_{2^u} = S_u = m_j$. Hence $[(\lambda_1, \lambda_2)_{2^p} = [(\lambda_1)_{2^p}, (\lambda_2)_{2^p}] = [m_k, m_j]$ and $y = [\lambda_1, \lambda_2]$ satisfies (1).

We next show that $\sigma(\lambda_1) \cap \sigma(\lambda_2) \subseteq \{2^p\}$. Suppose that $b \in \sigma(\lambda_1) \cap \sigma(\lambda_2)$ and observe that $\sigma(\lambda_i) = \{v - q_i | v \in \sigma(h_1w)\}$, for $i = 1, 2$. It follows that $b = v_1 - q_1 = \ldots = v_2 - q_2$. Therefore $b \in \{2^p\}$, which completes the proof.
v_2 - q_2 \quad \text{where} \quad v_1, v_2 \in \sigma(h_1 w). \quad \text{Therefore} \quad v_2 = v_1 - q_1 + q_2 = v_1 - 2r + 2u \quad \text{and as} \quad t \quad \text{is the least integer in} \quad \sigma(h_1 w) \quad \text{this implies} \quad v_2 \geq t - 2r + 2u. \quad \text{However} \quad 2r - t < 2u - 1, \quad \text{so} \quad v_2 > 2u - 1 \geq N. \quad \text{It follows from our choice of} \quad N \quad \text{and the fact that} \quad v_2 \in \sigma(h_1 w) \quad \text{that} \quad v_2 = 2^{n_2} \quad \text{for some integer} \quad n_2 \geq u. \quad \text{Now} \quad v_1 = 2^{n_2} + q_1 - q_2 = 2^{n_2} + 2r - 2u \geq 2r \geq N \quad \text{so} \quad v_1 = 2^{n_1} \quad \text{for some integer} \quad n_1 \geq r. \quad \text{Hence} \quad b = 2^{n_1} + 2p - 2r = 2^{n_2} + 2p - 2u \quad \text{so} \quad 2r(2^{n_1 - r} - 1) = 2r(2^{n_2 - r} - 2u - r) \quad \text{which implies that} \quad n_1 = r \quad \text{and} \quad n_2 = u. \quad \text{Thus} \quad b = 2^p \quad \text{and} \quad \sigma(\lambda_1) \cap \sigma(\lambda_2) \subseteq \{2^p\}. \quad \text{Note that} \quad \sigma([\lambda_1, \lambda_2]) \subseteq \sigma(\lambda_1) \cap \sigma(\lambda_2) \quad \text{so} \quad ([\lambda_1, \lambda_2])_l = 1 \quad \text{if} \quad l \neq 2^p \quad \text{and} \quad y = [\lambda_1, \lambda_2] \quad \text{satisfies} \quad (2) \quad \text{The proof is complete.}

Lemma 3.1 has the following easy consequence.

**LEMMA 3.2.** The subgroup $H$ of $G_M$ is omissible.

**PROOF.** Suppose $G_M = \langle H, X \rangle$ where $X$ is a subset of $G_M$. Define $N = \langle X \rangle$ so that $G = HN$. Now there exists $h_1, h_2 \in H$ and $n_1, n_2 \in N$ such that $w = h_1^{-1}n_1$ and $z = h_2^{-1}n_2$. Thus $h_1w, h_2z \in N$, so $N = G_M$, by Lemma 3.1. Therefore $N = G_M = \langle X \rangle$, as required.

Of course, Lemma 3.2 implies that $H \leq \Phi(G_M)$.

Our present goal, as mentioned at the end of §2, is to show that $\Phi(G_M) = H$. We have established that $\Phi(G_M) \geq H$, so it is enough to prove that $\Phi(G_M/H) = 1$. A routine argument, along the lines of [1, Theorem 7], and using the fact that $M$ is of infinite exponent, shows that $G_M/H \simeq Z \wr Z$, the restricted wreath product of two infinite cyclic groups. Moreover, it is easily seen that the Frattini subgroup of $Z \wr Z$ is trivial. Summing up, we have

**LEMMA 3.3.** Let $M$ be a countable perfect group of infinite exponent and let $G_M$ be the group constructed above. Then $\Phi(G_M) = H$, a direct product of $\aleph_0$ copies of $M$.

**4. Proof of Theorem 1.1.** If $K$ is of infinite exponent the result follows by Lemma 3.3, so suppose $K$ is of finite exponent.

Let $N$ be a countable perfect group of infinite exponent and define $M = K \times N$, a countable perfect group of infinite exponent. Consider the group $G_M$ constructed above. By Lemma 3.3 we have $\Phi(G_M) = H = \text{Dr}_{i \in Z} M_i$, where $M_i \simeq M$ and $M_i^z = M_{i+j}$, for all $i, j \in Z$.

We may write $M_0$ as a direct product $M_0 = K_0 \times N_0$, where $K_0 \simeq K$ and $N_0 \simeq N$. Let $K_i = K_0^Z$, $N_i = N_0^Z$, and notice that $M_i = K_i \times N_i$, $N_i^z = N_{i+1}$, and $N_i^w = N_i$, for all $i \in Z$. Lemma 3.1 shows that $G_M = \langle w, z \rangle$ and it follows that $D = \text{Dr}_{i \in Z} N_i$ is a normal subgroup of $G_M$.

Clearly $H/D \simeq \text{Dr}_{i \in Z} K_i$ and the result follows on setting $G_K = G_M/D$.

**ACKNOWLEDGMENTS.** The results presented here are contained in Chapter V of my Ph.D. thesis. I would like to record my gratitude to my supervisor, James Wiegold, who gave me much help and encouragement during my time as a research student. I am also indebted to the University of Wales for providing me with financial support.
REFERENCES


Department of Mathematics, University of Alabama, University, Alabama 35486