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# TWO-GENERATOR GROUPS WITH PERFECT FRATTINI SUBGROUPS

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ABSTRACT. In this paper, 2-generator groups with perfect Frattini subgroups are constructed. The construction also shows that every countable group can be embedded in the Frattini subgroup of a 2-generator group.

**1. Introduction.** It is well-known that the Frattini subgroup $\Phi(G)$ of a finite group $G$ is nilpotent. However, for infinite groups the situation is considerably more complicated. For instance, in [**1**] P. Hall constructed infinite 2-generator soluble groups with non-nilpotent Frattini subgroups. More recently, M. F. Newman (unpublished) has constructed finitely generated, infinite $p$-groups with insoluble Frattini subgroups.

In this note we use a construction similar to Hall's (loc. cit.) to obtain 2-generator groups with perfect Frattini subgroups. More precisely, we prove

THEOREM 1.1. *Let $K$ be a countable perfect group. Then there exists a 2-generator group $G_K$, such that $\Phi(G_K)$ is a direct product of $\aleph_0$ copies of $K$.*

Since each countable group can be embedded in a countable, nonabelian, simple (and therefore perfect) group (e.g. [**2**, Theorem 3.4]), Theorem 1.1 implies the following.

COROLLARY 1.2. *Every countable group can be embedded in the Frattini subgroup of a 2-generator group.*

**2. The construction.** Throughout the sequel $M$ denotes a countable perfect group of *infinite exponent.*

Suppose $M = \{m_1, m_2, m_3, \dots\}$ and let $S$ denote the sequence $m_1, m_2, m_1, m_2, m_3, m_1, m_2, m_3, m_4, m_1, \dots, m_i, m_1, \dots, m_i, m_{i+1}, m_1, \dots$ . Write $S_j$ for the $j$th term of $S$.

REMARK. Note that for any integer $N$ and any $m \in M$ there exists an integer $n > N$ such that $S_n = m$. This fact will be crucial in the proof of Lemma 3.1.

Let $C = \mathrm{Cr}_{i\in\mathbf{Z}} M_i$ and $H = \mathrm{Dr}_{i\in\mathbf{Z}} M_i$, where $M_i \simeq M$ for all $i \in \mathbf{Z}$. Thus $C$ is the cartesian product and $H$ the direct product of $\aleph_0$ copies of $M$. Given any $g \in C$ we write $(g)_i$ for the $i$th coordinate of $g$ and define the *support* of $g$ to be the set $\sigma(g) = \{i | (g)_i \neq 1\}$. Clearly $g \in H$ if and only if $\sigma(g)$ is a finite set.

Let $z$ generate an infinite cyclic group that is disjoint from $C$ and define an action of $\langle z\rangle$ on $C$ by

$$(g^z)_{i+1} = (g)_i, \quad \text{for all } g \in C \text{ and } i \in \mathbf{Z}.$$

Define $C^*$ to be the split extension of $C$ by $\langle z\rangle$.

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Let $w \in C$ be the element such that

$$(w)_{2^j} = S_j \quad \text{if } j \geq 1,$$
$$(w)_l = 1 \quad \text{if } l \neq 2^j \text{ for all } j \geq 1.$$

Finally, define $G_M = \langle H, w, z \rangle$, a subgroup of $C^*$. Observe that $M_i^z = M_{i+1}$ and $M_i^w = M_i$, for all $i \in \mathbf{Z}$, so $H \lhd G_M$.

The notation introduced above will be used for the remainder of this note. We shall show that $\Phi(G_M) = H$ and use this fact to deduce Theorem 1.1.

**3. The lemmas.** Our main lemma is the following.

LEMMA 3.1. *Let $h_1$ and $h_2$ be arbitrary elements of $H$. Then $G_M = \langle h_1 w, h_2 z \rangle$.*

PROOF. Since $\sigma(h_1)$ and $\sigma(h_2)$ are finite sets, there exists an integer $N > 0$ such that $(h_1 w)_n = (w)_n$ and $(h_2)_n = 1$ for all $n \geq N$. We consider such an $N$ fixed once for all and define $p$ to be the least integer such that $2^p \geq N$. We shall prove that $M_{2^p} \leq \langle h_1 w, h_2 z \rangle$. Since $(h_2 z)^{-k} M_{2^p} (h_2 z)^k = M_{2^p + k}$, for all $k \in \mathbf{Z}$, and $H = \mathrm{Dr}_{i \in \mathbf{Z}} M_i$, the result then follows immediately.

As $M$ is perfect, it suffices to show that, given arbitrary elements $m_k, m_j \in M$, there exists $y \in \langle h_1 w, h_2 z \rangle \cap H$ such that

$$(y)_{2^p} = [m_k, m_j] \tag{1}$$

and

$$(y)_l = 1 \quad \text{if } l \neq 2^p. \tag{2}$$

Suppose that $m_k$ and $m_j$ are given elements of $M$ and let $t$ denote the least integer in $\sigma(h_1 w)$. It follows from the remark in §2 that there exists an integer $r > p$ such that $S_r = m_k$. Furthermore, there exists an integer $u > r$ such that $S_u = m_j$ and $2^r - t < 2^{u-1}$. The letters $r$ and $u$ retain this meaning throughout.

Define $\lambda_1, \lambda_2 \in \langle h_1 w, h_2 z \rangle$ by $\lambda_i = (h_2 z)^{q_i} h_1 w (h_2 z)^{-q_i}$, where $q_1 = 2^r - 2^p$ and $q_2 = 2^u - 2^p$. We claim that $y = [\lambda_1, \lambda_2]$ is the desired element satisfying (1) and (2).

Bearing in mind that $q_i > 0$, for $i = 1, 2$, we have

$$(h_2 z)^{q_i} = h_2 h_2^{z^{-1}} \cdots h_2^{z^{-(q_i - 1)}} z^{q_i} = f_i z^{q_i}, \quad \text{say},$$

where $f_i \in H$. Thus $(\lambda_i)_{2^p} = (f_i)_{2^p} ((h_1 w)^{z^{-q_i}})_{2^p} (f_i^{-1})_{2^p}$. However, $f_i = \prod_{a=0}^{q_i - 1} h_2^{z^{-a}}$ so

$$(f_i)_{2^p} = \prod_{a=0}^{q_i - 1} (h_2^{z^{-a}})_{2^p} = \prod_{a=0}^{q_i - 1} (h_2)_{2^p + a}$$

for $i = 1, 2$. Now $2^p + a \geq N$ for $a = 0, 1, \ldots, q_i - 1$ and, by our choice of $N$, $(h_2)_n = 1$ for all $n \geq N$. Hence $(f_i)_{2^p} = 1$ for $i = 1, 2$. Therefore $(\lambda_i)_{2^p} = ((h_1 w)^{z^{-q_i}})_{2^p} = (h_1 w)_{2^p + q_i}$ for $i = 1, 2$. Recall that $(h_1 w)_n = (w)_n$ for all $n \geq N$. It follows that $(\lambda_1)_{2^p} = (h_1 w)_{2^r} = (w)_{2^r} = S_r = m_k$ by our choice of $r$, and similarly $(\lambda_2)_{2^p} = (h_1 w)_{2^u} = S_u = m_j$. Hence $([\lambda_1, \lambda_2])_{2^p} = [(\lambda_1)_{2^p}, (\lambda_2)_{2^p}] = [m_k, m_j]$ and $y = [\lambda_1, \lambda_2]$ satisfies (1).

We next show that $\sigma(\lambda_1) \cap \sigma(\lambda_2) \subseteq \{2^p\}$. Suppose that $b \in \sigma(\lambda_1) \cap \sigma(\lambda_2)$ and observe that $\sigma(\lambda_i) = \{v - q_i | v \in \sigma(h_1 w)\}$, for $i = 1, 2$. It follows that $b = v_1 - q_1 =$

$v_2 - q_2$ where $v_1, v_2 \in \sigma(h_1 w)$. Therefore $v_2 = v_1 - q_1 + q_2 = v_1 - 2^r + 2^u$ and as $t$ is the least integer in $\sigma(h_1 w)$ this implies that $v_2 \geq t - 2^r + 2^u$. However $2^r - t < 2^{u-1}$, so $v_2 > 2^{u-1} \geq N$. It follows from our choice of $N$ and the fact that $v_2 \in \sigma(h_1 w)$ that $v_2 = 2^{n_2}$ for some integer $n_2 \geq u$. Now $v_1 = 2^{n_2} + q_1 - q_2 = 2^{n_2} + 2^r - 2^u \geq 2^r \geq N$ so $v_1 = 2^{n_1}$ for some integer $n_1 \geq r$. Hence $b = 2^{n_1} + 2^p - 2^r = 2^{n_2} + 2^p - 2^u$ so $2^r(2^{n_1 - r} - 1) = 2^r(2^{n_2 - r} - 2^{u-r})$ which implies that $n_1 = r$ and $n_2 = u$. Thus $b = 2^p$ and $\sigma(\lambda_1) \cap \sigma(\lambda_2) \subseteq \{2^p\}$.

Note that $\sigma([\lambda_1, \lambda_2]) \subseteq \sigma(\lambda_1) \cap \sigma(\lambda_2)$ so $([\lambda_1, \lambda_2])_l = 1$ if $l \neq 2^p$ and $y = [\lambda_1, \lambda_2]$ satisfies (2). The proof is complete.

Lemma 3.1 has the following easy consequence.

LEMMA 3.2. *The subgroup $H$ of $G_M$ is omissible.*

PROOF. Suppose $G_M = \langle H, X \rangle$ where $X$ is a subset of $G_M$. Define $N = \langle X \rangle$ so that $G = HN$. Now there exists $h_1, h_2 \in H$ and $n_1, n_2 \in N$ such that $w = h_1^{-1} n_1$ and $z = h_2^{-1} n_2$. Thus $h_1 w, h_2 z \in N$, so $N = G_M$, by Lemma 3.1. Therefore $N = G_M = \langle X \rangle$, as required.

Of course, Lemma 3.2 implies that $H \leq \Phi(G_M)$.

Our present goal, as mentioned at the end of §2, is to show that $\Phi(G_M) = H$. We have established that $\Phi(G_M) \geq H$, so it is enough to prove that $\Phi(G_M/H) = 1$. A routine argument, along the lines of [**1**, Theorem 7], and using the fact that $M$ is of infinite exponent, shows that $G_M/H \simeq \mathbf{Z} \operatorname{wr} \mathbf{Z}$, the restricted wreath product of two infinite cyclic groups. Moreover, it is easily seen that the Frattini subgroup of $\mathbf{Z} \operatorname{wr} \mathbf{Z}$ is trivial. Summing up, we have

LEMMA 3.3. *Let $M$ be a countable perfect group of infinite exponent and let $G_M$ be the group constructed above. Then $\Phi(G_M) = H$, a direct product of $\aleph_0$ copies of $M$.*

**4. Proof of Theorem 1.1.** If $K$ is of infinite exponent the result follows by Lemma 3.3, so suppose $K$ is of finite exponent.

Let $N$ be a countable perfect group of infinite exponent and define $M = K \times N$, a countable perfect group of infinite exponent. Consider the group $G_M$ constructed above. By Lemma 3.3 we have $\Phi(G_M) = H = \operatorname{Dr}_{i \in \mathbf{Z}} M_i$, where $M_i \simeq M$ and $M_i^{z^j} = M_{i+j}$, for all $i, j \in \mathbf{Z}$.

We may write $M_0$ as a direct product $M_0 = K_0 \times N_0$, where $K_0 \simeq K$ and $N_0 \simeq N$. Let $K_i = K_0^{z^i}$, $N_i = N_0^{z^i}$, and notice that $M_i = K_i \times N_i$, $N_i^z = N_{i+1}$, and $N_i^w = N_i$, for all $i \in \mathbf{Z}$. Lemma 3.1 shows that $G_M = \langle w, z \rangle$ and it follows that $D = \operatorname{Dr}_{i \in \mathbf{Z}} N_i$ is a normal subgroup of $G_M$.

Clearly $H/D \simeq \operatorname{Dr}_{i \in \mathbf{Z}} K_i$ and the result follows on setting $G_K = G_M/D$.

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